Note on phase diagrams

MA18Q1-H

mail@**kenjisato**.jp

April 24, 2018

1 Phase diagram analysis

For a macroeconomic model with two variables, phase diagram analysis is an important analytic tool. Here is a step-by-step instruction. Suppose that the dynamics of a macro model is determined by

$$\begin{aligned}\dot{x} &= f(x, y)\\ \dot{y} &= g(x, y),\end{aligned}$$

where x and y are the stock variables of interest (for example, physical and human capital per unit of effective labor, k and h).

Step 1: Draw a (x, y)-space

Draw x-axis horizontally and y-axis vertically.

Step 2: Draw graphs for $f(x, y) = 0$ and $g(x, y) = 0$.

Since $f(x, y) = 0$ holds if and only if $\dot{x} = 0$, the graph corresponds to the region in which there is no instantaneous movement in x. Same for the graph of $g(x, y) = 0$. Note that I said "instantaneous" movement in x. That does not mean a steady state. If $\dot{x} = 0$, there is no movement parallel to the x-axis, but there may be a movement parallel to the y-axis. The vertical movement shifts (x, y) apart from the graph of $f(x, y) = 0$ and$x$ begins to move again.

At the intersection of $f(x, y) = 0$ and $g(x, y) = 0$, $\dot{x} = 0$ and $\dot{y} = 0$ hold simultaneously. This is the steady state.

Step 3: Determine the direction of movement and write in arrows.

The two graphs draw several (typically, four) divisions in the first quadrant in the (x, y)-space. In each division, one of the following four conditions hold.

1. $\dot{x} > 0\ (\rightarrow)$ and $\dot{y} > 0\ (\uparrow)$
2. $\dot{x} > 0\ (\rightarrow)$ and $\dot{y} < 0\ (\uparrow)$
3. $\dot{x} < 0\ (\leftarrow)$ and $\dot{y} > 0\ (\downarrow)$
4. $\dot{x} < 0\ (\leftarrow)$ and $\dot{y} < 0\ (\downarrow)$

Step 4: Simulate the dynamics from a given initial state

Pick several initial points. Draw a trajectory from each point, carefully following the directions indicated by the arrows.

Step 5: Observe stability of the steady state

The steady state is stable, saddle-point stable or unstable. If it is stable, the trajectory converges to the steady state regardless of the steady state. If it is unstable, the trajectory cannot converge to the steady state. If it is saddle-point stable, you must carefully choose an initial state to find a convergent trajectory.

2 An exercise

Graphically analyze the dynamics of the following system of differential equations.

$$\begin{aligned} \dot{x} &= x - y + 1 \\ \dot{y} &= -x - y + 2 \end{aligned}$$

Is the steady state stable, saddle-point stable or unstable? If it is saddle-point stable, where are the initial points starting from which the trajectory approches to the steady state?