Problem Set

#02

mail@**kenjisato**.jp

Day 2

[1] Differential equations

In each case below, find a differential equation that describes the development of a stock variable.

1. The rates of withdrawal, W, and deposit, D, determine the bank balance, or savings, S.
2. The rates of birth, B, and death, D, determines the population, N.
3. The rates of government expenditure, G, tax receipts, T, and interest rate r determine the govenment debt, D.

[2] Prediction for the near future

Suppose that your clock shows it is time t now. Suppose also that some important variable $k(t)$ satisfies a differential equation $\dot{k} = f(k)$, where $\dot{k}$ denotes the time derivative of k, f is a function of k. You are interested in prediction of $k(t + \Delta t)$ for a sufficiently small $\Delta t > 0$.

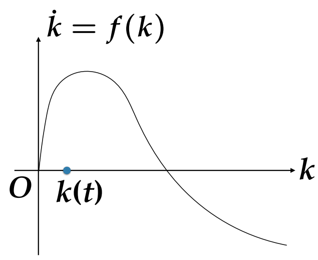
In the cases listed below, is $k(t + \Delta t)$ greater than, smaller than or equal to $k(t)$?

1. $f(k(t)) > 0$
2. $f(k(t)) < 0$
3. $f(k(t)) = 0$

[3] Prediction for the distant future

Consider a differential equation $\dot{k} = f(k)$. The below figure sketches the graph of f.

Describe what will happen to $k(t)$ as $t \to \infty$, starting from $k(t)$ indicated by the dot.



Notation

log or ln

Both log and ln denote the natural logarithm. It is the inverse function of e^x, where $e \simeq 2.718281...$ is Napier's constant; i.e., $e^{\ln x} = x$, and $\ln e^x = x$. When we want to specify the base b of log, we explicitly write it. For instance, the common logarithm is denoted by $\log_{10} y$.

We will use the following formulas very often: for $x, y > 0$,

$$\ln xy = \ln x + \ln y, \quad \ln \frac{x}{y} = \ln x - \ln y, \quad \ln x^\alpha = \alpha \ln x.$$

Derivatives

Suppose a variable x changes its values with time. Since we usually use letter t for time, we write $x(t)$ to show it is time dependent; it is a function of time. We sometimes don't bother to write t when the time-dependence is obvious from the context. The derivative of x with respect to time

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t}$$

is denoted by $\dot{x}(t)$. $\dot{x}$ is voiced "x dot."

Let f be a function of x. The derivative of $f(x)$ with respect to x is denoted by $f'(x)$ (f' is voiced "f prime"). When f is a function of x and x is a function of time, the most unambiguous expression, $f(x(t))$, is sometimes written simply as $f(x)$. Because $f(x)$ is a function of time (through the time dependence of x), we can take time-drivative of $f(x)$, which is

$$\frac{df(x)}{dt} = f'(x(t))\dot{x}(t).$$

For example, let $f = \ln$. Recall that $f'(x) = \ln'(x) = \frac{1}{x}$. Thus,

$$\frac{d}{dt}(\ln x(t)) = \frac{\dot{x}(t)}{x(t)}.$$

Growth Rates

Since we will study economic growth, we will analyze the rates of growth of many economic variables. Mathematically, the growth rate of x is defined by $\frac{\dot{x}}{x}$, which will be denoted by g_x in this class (this is not standard). Recall that

$$\frac{\dot{x}}{x} \simeq \frac{x(t + \Delta t) - x(t)}{\Delta t \cdot x(t)} = \frac{1}{\Delta t}\left[\frac{x(t + \Delta t) - x(t)}{x(t)}\right].$$

By multiplying $\frac{1}{\Delta t}$ and the instantaneous rate of change, $\frac{x(t+\Delta t) - x(t)}{x(t)}$, the latter is translated up into the rate of change in a unit length of time, a year, quarter, or month for instance.