– an invitation –

Exploring hypercomputation with the effective topos

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1 Crash course on ordinal numbers

2 (Super) Turing machines
- Bacics on Turing machines
- Bacis on super Turing machines
- The power of super Turing machines
- Outlook on the larger theory

3 The effective topos
- First steps in the effective topos
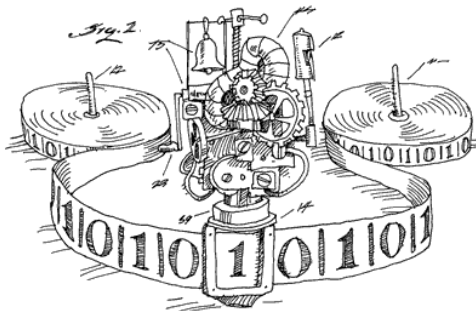- Curious size phenomena
- Wrapping up

Part I

A crash course on ordinal numbers

Part II

(Super) Turing machines

Basics on Turing machines

- Turing machines are idealised computers operating on an **infinite tape** according to a **finite list** of rules.
- The concept is astoundingly robust.
- A subset of $\mathbb{N}$ is **enumerable by a Turing machine** if and only if it's a Σ_1-set.

Alan Turing
(* 1912, † 1954)

worth watching

Alison Bechdel
(* 1960)

Super Turing machines

With super Turing machines, the time axis is more interesting:

- normal: 0, 1, 2, ...
- super: 0, 1, 2, ..., ω, $\omega + 1$, ..., $\omega \cdot 2$, $\omega \cdot 2 + 1$, ...

On reaching a limit ordinal time step like ω or $\omega \cdot 2$, ...

Super Turing machines
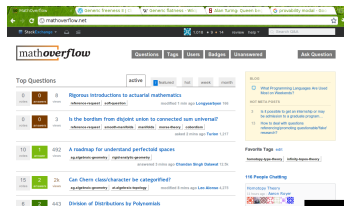
With super Turing machines, the time axis is more interesting:

- normal: $0, 1, 2, \ldots$
- super: $\quad 0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots\ldots\ldots$

On reaching a limit ordinal time step like ω or $\omega \cdot 2, \ldots$

- the machine is put into a designated state,
- the read/write head is moved to the start of the tape, and
- the tape is set to the "lim sup" of all its previous contents.

Joel David Hamkins   MathOverflow   Andy Lewis

A question for you

What's the behaviour of this super Turing machine?

In the start state and the limit state, check whether
the current cell contains a "1".

- If yes, then stop.
- If not, then flash that cell: set it to "1", then
  reset it to "0". Then unremittingly move the
  head rightwards.

A question for you

What's the behaviour of this super Turing machine?

> In the start state and the limit state, check whether the current cell contains a "1".
>
> - If yes, then stop.
> - If not, then flash that cell: set it to "1", then reset it to "0". Then unremittingly move the head rightwards.

This machine halts after time step ω^2.

Super Turing machines can break out of (some kinds of) infinite loops.

What can super Turing machines do?

- Everything ordinary Turing machines can do.
- Verify number-theoretic statements.
- Decide whether a given ordinary Turing machine halts.
- Simulate super Turing machines.
- Decide Π_1^1- and Σ_1^1-statements:
    - "For every function $\mathbb{N} \to \mathbb{N}$ it holds that …"
    - "There is a function $\mathbb{N} \to \mathbb{N}$ such that …"
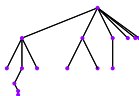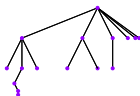
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- Decide Π_1^1- and Σ_1^1-statements:
  - "For every function $\mathbb{N} \to \mathbb{N}$ it holds that ..."
  - "There is a function $\mathbb{N} \to \mathbb{N}$ such that ..."

But: Super Turing machines can't compute all functions and can't write every 0/1-sequence to the tape.

Fun facts

- Every super Turing machine either halts or gets caught in an unbreakable infinite loop after **countably many steps**.

- An ordinal number α is **clockable** iff there is a super Turing machine which halts precisely after time step α.
  - Speed-up Lemma: If $\alpha + n$ is clockable, then so is α.
  - Big Gaps Theorem
  - Many Gaps Theorem
  - Gapless Blocks Theorem

- **Lost Melody Theorem:** There are 0/1-sequences which a super Turing machine can recognise, but not write to the tape.

Part III

The effective topos

The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 Beyond every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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2 Beyond every number there is a prime.	✔	✔	✔
3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✔ (trivially)	✗	✔
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✔ (trivially)	?	?
7 Countable choice holds.	✔	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ **1**" amounts to: There is a Turing machine which determines of any given number whether it is prime or not.

The effective topos

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1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 Beyond every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ **2**" amounts to: There is a Turing machine which, given a number n, computes a prime larger than n.

The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 Beyond every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	？	？
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	？	？
6 Markov's principle holds.	✓ (trivially)	？	？
7 Countable choice holds.	✓	？	？
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	？	？

"Eff(TM) $\models$ **3**" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, determines whether f has a zero or not.

The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 Beyond every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	?
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ **4**" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs a Turing machine computing f.

The effective topos

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2 Beyond every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	❓	❓
6 Markov's principle holds.	✓ (trivially)	❓	❓
7 Countable choice holds.	✓	❓	❓
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	❓	❓

"Eff(STM) $\models$ **4**" amounts to: There is a super Turing machine which, given a *super* Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs an (*ordinary*) Turing machine computing f.

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3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	？
6 Markov's principle holds.	✓ (trivially)	？	？
7 Countable choice holds.	✓	？	？
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	？	？

A real number of Eff(TM) is externally represented by a Turing machine M which on input n outputs a rational approximation $M(n)$. These approximations need to be compatible in that $|M(n) - M(m)| \leq 2^{-n} + 2^{-m}$ for all n, m.

Two such machines M and M' represent the same real number iff $|M(n) - M'(m)| \leq 2^{-n} + 2^{-m}$ for all n, m.

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4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
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4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ **6**" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$ and given the promise that it is *not not* the case that f has a zero, determines a zero of f.

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3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7 Countable choice holds.	✓	?	?
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4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7 Countable choice holds.	✓	✓ (always!)	？
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	？	？

"Eff(TM) $\models$ **7**" amounts to: There is a Turing machine which, given a Turing machine computing for every $x \in \mathbb{N}$ some $y \in A$ together with a witness of $\varphi(x, y)$, outputs a Turing machine computing a suitable choice function $\mathbb{N} \to A$.

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5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
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8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	？	？

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4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
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7 Countable choice holds.	✓	✓ (always!)	✓ (always!)
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	✗	✓

Curious size phenomena

$\mathrm{Eff}(\mathrm{STM}) \models$ "There exists an injection $\mathbb{R} \hookrightarrow \mathbb{N}$."

means:

> There is a super Turing machine which inputs the source of a super Turing machine A representing a real number and outputs a natural number $n(A)$ such that $n(A) = n(B)$ if and only if A and B represent the same real.

Curious size phenomena

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This statement is witnessed by following super Turing machine:

> Read the source of a super Turing machine A from the tape. Simulate all super Turing machines in a dovetailing fashion. As soon a machine is found which represents the same real as A, output the index of this machine and halt.

Wrapping up

- Effective toposes are a good vehicle for studying the nature of computation.

- Effective toposes build links between constructive mathematics and programming.

- Toposes allow for curious dream axioms.

- Toposes also have a geometric flavour: points, subtoposes, continuous maps between toposes.
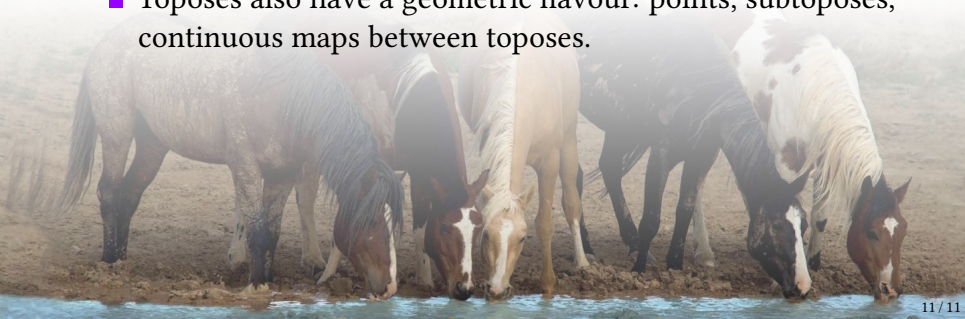
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There is more to mathematics than the standard topos.