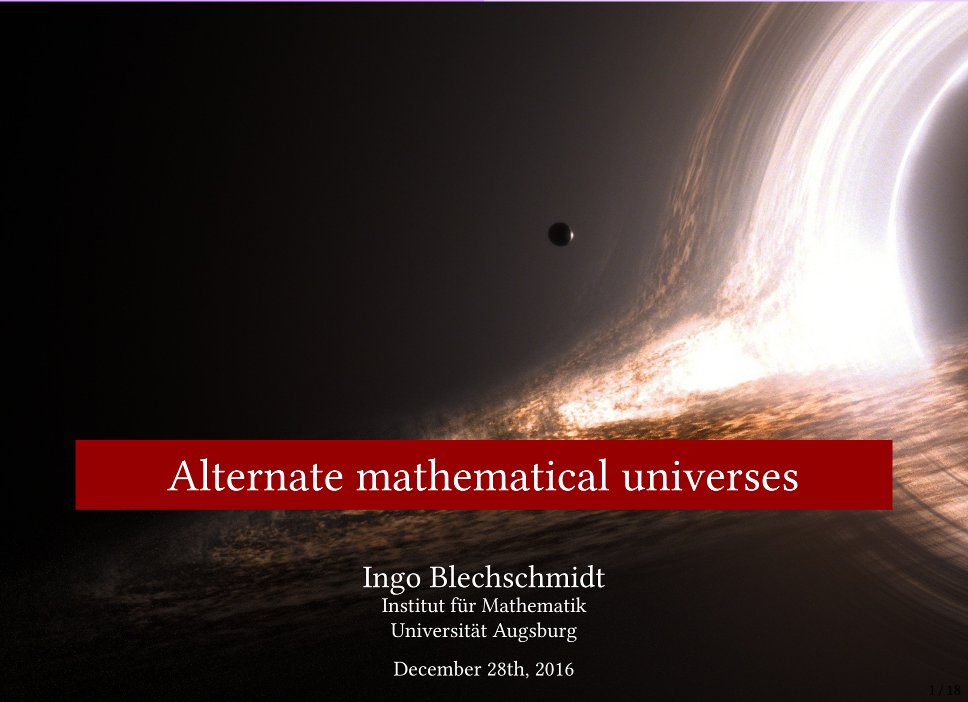

Alternate mathematical universes

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On truth

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"There are exactly five platonic solids."

"There are more real numbers than integers."

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- Mathematical statements are thought to be absolutely objective and independent of human sentiments.
- But in fact, their truth depends on the **choice of mathematical universe** ("topos").
- The usual axioms of logic hold in the **standard topos** Set.
- In other topoi, slightly different axioms are true.

Examples

There are topoi in which …

- there is an intermediate infinity between $\mathbb{N}$ and $\mathbb{R}$,
- there is no such intermediate infinity,
- there are **infinitesimal real numbers** ε such that $\varepsilon \neq 0$ but $\varepsilon^2 = 0$, or
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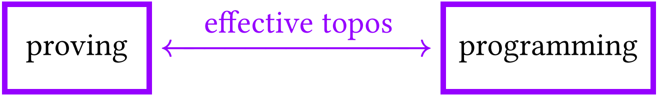


The effective topos

- "$1 + 1 = 2$."
  True in Set, true in $\mathrm{Eff}(\mathrm{TM})$.

- "Any number is either prime or not."
  Trivially true in Set, nontrivially true in $\mathrm{Eff}(\mathrm{TM})$.

- "Any function $\mathbb{N} \to \mathbb{N}$ is either the zero function or not."
  Trivially true in Set, false in $\mathrm{Eff}(\mathrm{TM})$.

- "Any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine."
  False in Set, trivially true in $\mathrm{Eff}(\mathrm{TM})$.

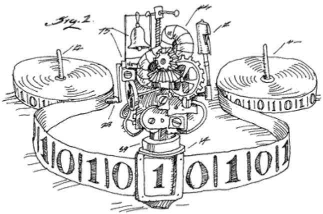
- "Any function $\mathbb{R} \to \mathbb{R}$ is continuous."
  False in Set, true in $\mathrm{Eff}(\mathrm{TM})$.

The effective topos

There are many **models of computation**:

- Turing machines
- Lambda calculus
- Perl programs (running on idealized computers)
- Computers in the real physical world

For any such model $\mathcal{M}$, there is a special topos $\mathrm{Eff}(\mathcal{M})$, the associated **effective topos**.



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$\mathrm{Eff}(\mathcal{M}) \models$ "For any number n there is a prime $p > n$."

means:

There is a program which reads a number n as input and outputs a prime number $p > n$.

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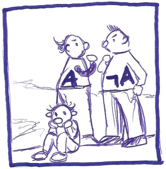
$\mathrm{Eff}(\mathcal{M}) \models$ "Any number is either prime or not prime."
means:
There is a program which reads a number n as input and outputs YES or NO depending on whether n is prime or not.

What's true in alternate topoi?

Metatheorem: If a statement has a **constructive proof**, then it holds in **any topos**.

Constructive logic is like classical logic, except we don't suppose the **law of excluded middle** (LEM), which says:

- "Any statement is either true or not true."
- "If a statement is *not not* true, then it's true."



Nonconstructive proofs

A number is **rational** if and only if its decimal digits eventually repeat.

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Theorem. There are **irrational** numbers x und y such that x^y is rational.

Proof. Either $\sqrt{2}^{\sqrt{2}}$ is rational or not.

1. In the first case we are done.
2. In the second case we set $x := \sqrt{2}^{\sqrt{2}}$ and $y := \sqrt{2}$. Then $x^y = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$ is rational.

Appreciating constructive logic

At first sight, dropping the axiom of excluded middle looks like a sad thing to do. It's a useful axiom! However:

- The axiom is not needed as often as one would think.
- The abstinence is good for your mental hygiene.
- Constructive logic allows for finer distinctions.
- From constructive proofs one can mechanically extract programs which witness the proved statements.
- **Dropping the law of excluded middle allows to add curious unconvential axioms.**

LEM for equality of functions

$\mathrm{Eff}(\mathrm{TM}) \models$ "Any function $f : \mathbb{N} \to \mathbb{N}$ is either the zero function or not."

means:

There is a program which reads the source of a program M, which calculates a function $\mathbb{N} \to \mathbb{N}$, as input, and finds out whether M always yields zero or not.

That's false.

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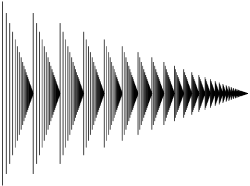
The statement is true in Eff(STM), the effective topos associated to **super Turing machines**.



LEM for the halting problem

$\mathrm{Eff}(\mathrm{TM}) \models$ "Any program halts or doesn't halt."

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is in constructive logic equivalent to

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so it's not true in $\mathrm{Eff}(\mathrm{TM})$, but in $\mathrm{Eff}(\mathrm{STM})$.

LEM for equality of real numbers

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so it's not true in Eff(TM), but in Eff(STM).

For a Turing machine M consider the real number $0.000\ldots$, whose n'th decimal digit is a one if and only if M halts after step n.

For a real number x consider the Turing machine which searches the digits of x for a nonzero digit.

Markov's principle

$\mathrm{Eff}(\mathrm{TM}) \models$ "For any function $f : \mathbb{N} \to \mathbb{N}$ which is not the zero function, there is a number $n \in \mathbb{N}$ such that $f(n) \neq 0$."

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That's true! By unbounded search.

Searching uncountable sets

> "For any function $f : \mathbb{N} \to \mathbb{N}$ from numbers to numbers, there either exists a number n such that $f(n)$ is one or there is no such number."

This statement is false in $\mathrm{Eff}(\mathrm{TM})$.

> "For any function $P : L(\mathbb{B}) \to \mathbb{B}$ from infinite lists of booleans to booleans, there either exists a list x such that $P(x)$ is true or there is no such list."

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The Church–Turing thesis

The **Church–Turing thesis** states: If a function $f : \mathbb{N} \to \mathbb{N}$ is calculable in the real world, then it's also calculable by a program (running on a Turing machine).

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That's trivial! "cat" is the looked-for program.

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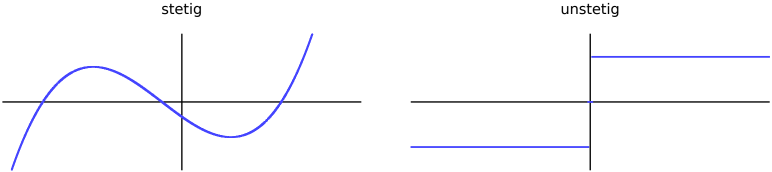
In $\mathrm{Eff}(\mathrm{STM})$ and $\mathrm{Eff}(\lambda\mathrm{C})$ the statement is false.

Automatic continuity

The following statement is wildly **false** in Set:

“Every function $f : \mathbb{R} \to \mathbb{R}$ is continuous.”

A function f is **continuous** if and only if, for calculating $f(x)$ to finitely many digits, finitely many digits of x suffice.



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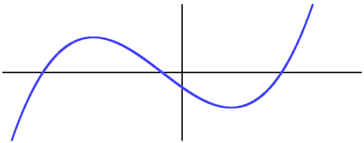

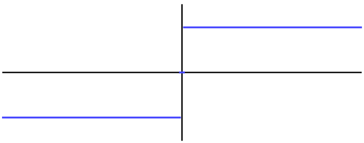

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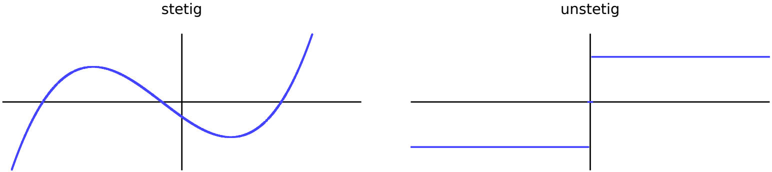

True in Eff(TM). True in Eff(RW), if private communication channels are possible and only finitely many computational steps can be executed in finite time.

Wrapping up

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- Topoi allow for curious dream axioms.
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There is more to mathematics than the standard topos.