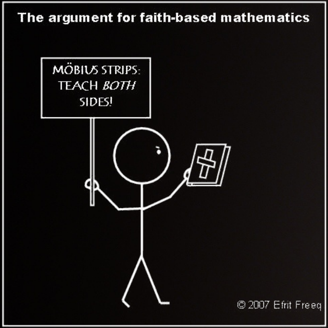


♥ Faith in mathematics ♥

34th Chaos Communication Congress

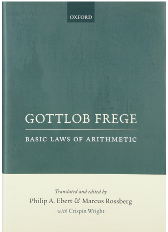
Questions are very much welcome! Please interrupt me mid-sentence.

Ingo Blechschmidt (University of Augsburg)

with thanks to Matthias Hutzler and Christian Ittner

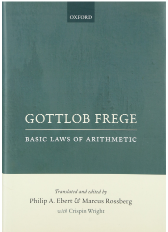
Part I

The foundational crisis in mathematics



Part I

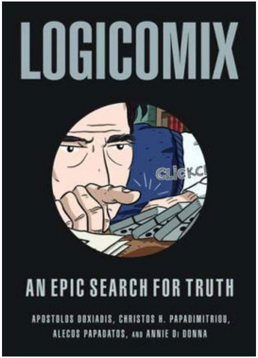
The foundational crisis in mathematics

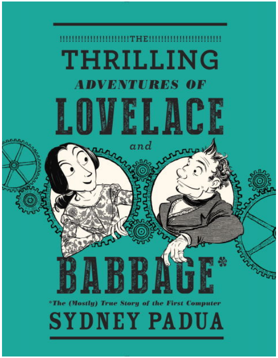


"Let U be the set of all those sets which don't contain themselves."
Naive mathematics is **inconsistent**, rendering it **unreliable**. ☹
Thus the **axiomatic method** was born.

Part I

The foundational crisis in mathematics





∗54·42. $\vdash :: \alpha \in 2 \,.\, \supset :. \beta \subset \alpha \,.\, \exists ! \beta \,.\, \beta \neq \alpha \,.\, \equiv \,.\, \beta \in \iota\text{“}\alpha$

Dem.

$\vdash .\, *54{\cdot}4 \,.\quad \supset \vdash :: \alpha = \iota\text{‘}x \cup \iota\text{‘}y \,.\, \supset :.$

$\beta \subset \alpha \,.\, \exists ! \beta \,.\, \equiv : \beta = \Lambda \,.\, \vee \,.\, \beta = \iota\text{‘}x \,.\, \vee \,.\, \beta = \iota\text{‘}y \,.\, \vee \,.\, \beta = \alpha : \exists ! \beta :$

[∗24·53·56.∗51·161] $\equiv : \beta = \iota\text{‘}x \,.\, \vee \,.\, \beta = \iota\text{‘}y \,.\, \vee \,.\, \beta = \alpha$ (1)

$\vdash .\, *54{\cdot}25 \,.\, \text{Transp} \,.\, *52{\cdot}22 \,.\, \supset \vdash : x \neq y \,.\, \supset \,.\, \iota\text{‘}x \cup \iota\text{‘}y \neq \iota\text{‘}x \,.\, \iota\text{‘}x \cup \iota\text{‘}y \neq \iota\text{‘}y :$

[∗13·12] $\supset \vdash : \alpha = \iota\text{‘}x \cup \iota\text{‘}y \,.\, x \neq y \,.\, \supset \,.\, \alpha \neq \iota\text{‘}x \,.\, \alpha \neq \iota\text{‘}y$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash :: \alpha = \iota\text{‘}x \cup \iota\text{‘}y \,.\, x \neq y \,.\, \supset :.$

$\beta \subset \alpha \,.\, \exists ! \beta \,.\, \beta \neq \alpha \,.\, \equiv : \beta = \iota\text{‘}x \,.\, \vee \,.\, \beta = \iota\text{‘}y :$

[∗51·235] $\equiv : (\exists z) \,.\, z \in \alpha \,.\, \beta = \iota\text{‘}z :$

[∗37·6] $\equiv : \beta \in \iota\text{“}\alpha$ (3)

$\vdash .\, (3) \,.\, *11{\cdot}11{\cdot}35 \,.\, *54{\cdot}101 \,.\, \supset \vdash \,.\, \text{Prop}$

∗54·43. $\vdash :. \alpha, \beta \in 1 \,.\, \supset : \alpha \cap \beta = \Lambda \,.\, \equiv \,.\, \alpha \cup \beta \in 2$

Dem.

$\vdash .\, *54{\cdot}26 \,.\, \supset \vdash :. \alpha = \iota\text{‘}x \,.\, \beta = \iota\text{‘}y \,.\, \supset : \alpha \cup \beta \in 2 \,.\, \equiv \,.\, x \neq y \,.$

[∗51·231] $\equiv \,.\, \iota\text{‘}x \cap \iota\text{‘}y = \Lambda \,.$

[∗13·12] $\equiv \,.\, \alpha \cap \beta = \Lambda$ (1)

$\vdash .\, (1) \,.\, *11{\cdot}11{\cdot}35 \,.\, \supset$

$\vdash :. (\exists x, y) \,.\, \alpha = \iota\text{‘}x \,.\, \beta = \iota\text{‘}y \,.\, \supset : \alpha \cup \beta \in 2 \,.\, \equiv \,.\, \alpha \cap \beta = \Lambda$ (2)

$\vdash .\, (2) \,.\, *11{\cdot}54 \,.\, *52{\cdot}1 \,.\, \supset \vdash \,.\, \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

∗54·44. $\vdash :. z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, \supset_{z,w} \,.\, \phi(z, w) : \equiv \,.\, \phi(x, x) \,.\, \phi(x, y) \,.\, \phi(y, x) \,.\, \phi(y, y)$

Dem.

$\vdash .\, *51{\cdot}234 \,.\, *11{\cdot}62 \,.\, \supset \vdash :. z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, \supset_{z,w} \,.\, \phi(z, w) : \equiv :$

$z \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, \supset_z \,.\, \phi(z, x) \,.\, \phi(z, y) :$

[∗51·234.∗10·29] $\equiv : \phi(x, x) \,.\, \phi(x, y) \,.\, \phi(y, x) \,.\, \phi(y, y) :. \supset \vdash \,.\, \text{Prop}$

∗54·441. $\vdash :: z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, z \neq w \,.\, \supset_{z,w} \,.\, \phi(z, w) : \equiv :. x = y : \vee : \phi(x, y) \,.\, \phi(y, x)$

Dem.

$\vdash .\, *5{\cdot}6 \,.\, \supset \vdash :: z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, z \neq w \,.\, \supset_{z,w} \,.\, \phi(z, w) : \equiv :.$

$z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, \supset_{z,w} : z = w \,.\, \vee \,.\, \phi(z, w) :.$

[∗54·44] $\equiv : x = x \,.\, \vee \,.\, \phi(x, x) : x = y \,.\, \vee \,.\, \phi(x, y) :$

$y = x \,.\, \vee \,.\, \phi(y, x) : y = y \,.\, \vee \,.\, \phi(y, y) :$

[∗13·15] $\equiv : x = y \,.\, \vee \,.\, \phi(x, y) : y = x \,.\, \vee \,.\, \phi(y, x) :$

[∗13·16.∗4·41] $\equiv : x = y \,.\, \vee \,.\, \phi(x, y) \,.\, \phi(y, x)$

This proposition is used in ∗163·42, in the theory of relations of mutually exclusive relations.

∗54·442. $\vdash :: x \neq y \,.\, \supset :. z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, z \neq w \,.\, \supset_{z,w} \,.\, \phi(z, w) : \equiv \,.\, \phi(x, y) \,.\, \phi(y, x)$ [∗54·441]

Part II

Truth and provability

A syntactic quality

A statement is **provable** if and only if it has a **formal proof** using only the **Peano axioms**.

Example. $1 + 1 = S(0) + S(0) = S(S(0) + 0) = S(S(0)) = 2.$

A semantic quality

A statement is **true** if and only if it holds in the **standard model**.

Mathematical induction and:

$$S(n) \neq 0 \qquad S(n) = S(m) \Rightarrow n = m$$
$$n + 0 = n \qquad n + S(m) = S(n + m)$$
$$n \cdot 0 = 0 \qquad n \cdot S(m) = n \cdot m + n$$

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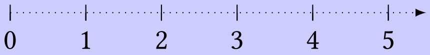

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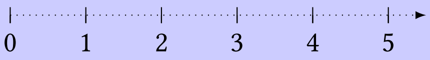
A semantic quality

A statement is **true** if and only if it holds in the **standard model**.

Provable statements are true.
True statements are **not** necessarily provable.



Part III

True but unprovable statements

For statements about **specific numbers**, there is no difference between provability and truth. But all of the following are unprovable:

- "This statement is not provable."
  But **take care**: Consider "This statement is not true".
- "Hercula can kill any hydra."
- "BB(9000) $= x$." (for the correct value x)
- "There is no proof of $1 = 0$." ("Peano arithmetic is consistent.")

Also: "There is an infinity/no infinity between $\mathbb{N}$ and $\mathbb{R}$."

Part IV

Fundamental incompleteness

Gödel discovered:

Any consistent and recursively axiomatizable formal system is **incomplete**.

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Fundamental incompleteness

Gödel discovered:

> **Any** consistent and recursively axiomatizable formal system is **incomplete**.

Going deeper:

> Peano arithmetic can **not** prove "Peano arithmetic is consistent".

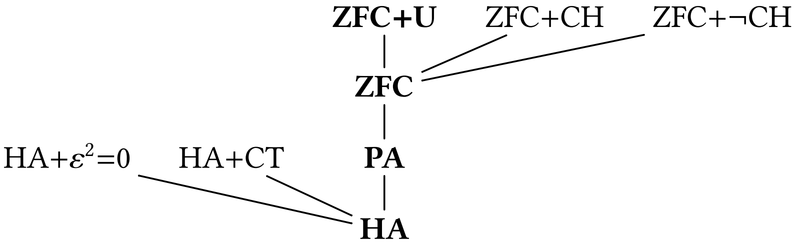
Proof idea: Get "this statement is not provable" to work.

- Express provability using numbers (think ASCII).
- Rewrite self-referentiality like this:
  "»yields an unprovable statement when preceded by its quotation« yields an unprovable statement when preceded by its quotation."

If the system is consistent, then that statement is true, but neither it nor its negation are provable.

Outlook

- We use the axiomatic method to make maths **reliable**.
- But any axiomatization is **incomplete**.



- If a statement holds in **all** models, it's provable.