1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50

Large numbers,
very large numbers,
very very large numbers

– an invitation to advanced googology –

Ingo Blechschmidt
Università di Verona

Matthias Hutzler
Universität Augsburg

35th Chaos Communication Congress
December 30th, 2018

Part 0

Large numbers

17 000 congress participants

$\mathbf{10^{19}} = \underbrace{\mathbf{10\,000\,000\,000\,000\,000\,000}}_{\text{19 zeros}}$ grains of sand on earth

$\mathbf{10^{80}} = \mathbf{1}\underbrace{\mathbf{000\ldots000}}_{\text{80 zeros}}$ elementary particles in the universe

Part I

Very large numbers

$$
\begin{aligned}
2 \cdot 4 &= 2 + 2 + 2 + 2 = 8 \\
2^4 &= 2 \cdot 2 \cdot 2 \cdot 2 = 16
\end{aligned}
$$

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Very large numbers

$$2 \cdot 4 = 2 + 2 + 2 + 2 = 8$$

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$$2 \uparrow\uparrow 4 = 2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65\,536$$

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2 \uparrow\uparrow\uparrow 4 &= 2 \uparrow\uparrow (2 \uparrow\uparrow (2 \uparrow\uparrow 2))
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&= 2 \uparrow\uparrow\uparrow 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} = \underbrace{2 \uparrow\uparrow (2 \uparrow\uparrow (2 \uparrow\uparrow (\cdots \uparrow\uparrow 2)))}_{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \text{ many two's}}
\end{aligned}$$

Part I

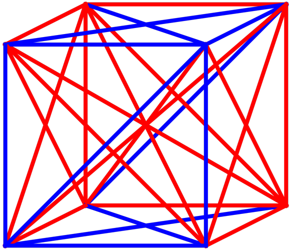
Very large numbers

$$\textbf{Graham's number} \quad = \quad \left.\begin{matrix} 3\underbrace{\uparrow \cdots\cdots\cdots\cdots\cdots\cdots \uparrow}3 \\ 3\underbrace{\uparrow \cdots\cdots\cdots\cdots\cdots \uparrow}3 \\ \vdots \\ \underbrace{\qquad\qquad\qquad\qquad} \\ 3\underbrace{\uparrow \cdots\cdots \uparrow}3 \\ 3 \uparrow\uparrow\uparrow\uparrow 3 \end{matrix}\right\} \text{64 layers}$$

Part I

Very large numbers

$$\textbf{Graham's number} \quad = \quad \left.\begin{array}{c} 3\uparrow\underbrace{\cdots\cdots\cdots\cdots\cdots\cdots}\uparrow 3 \\ 3\uparrow\underbrace{\cdots\cdots\cdots\cdots\cdots}\uparrow 3 \\ \vdots \\ \underbrace{\qquad\qquad\qquad\qquad} \\ 3\uparrow\underbrace{\cdots\cdots}\uparrow 3 \\ 3\uparrow\uparrow\uparrow\uparrow 3 \end{array}\right\}\text{64 layers}$$



$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}} = 2$$

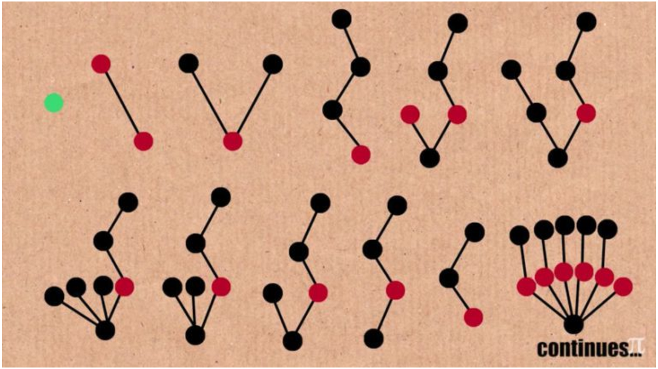
$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}} = 2$$

$$\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}} = 2$$

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \ldots$$

Part II

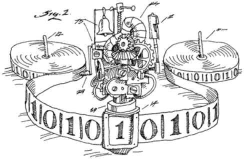
Very very large numbers



Any forest eventually dies, at a maximum of **TREE(3)** trees.

Part III

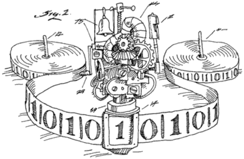
Very very very large numbers



- **BB(n)** is the maximal number of steps a Turing machine with n states can carry out before halting.

Part III

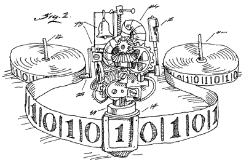
Very very very large numbers



- **BB(n)** is the maximal number of steps a Turing machine with n states can carry out before halting.
- The Busy Beaver function is **uncomputable**.

Part III

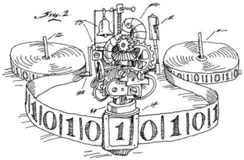
Very very very large numbers



- **BB(n)** is the maximal number of steps a Turing machine with n states can carry out before halting.
- The Busy Beaver function is **uncomputable** and **asymptotically dominates** any computable function.

Part III

Very very very large numbers



- **BB(n)** is the maximal number of steps a Turing machine with n states can carry out before halting.
- The Busy Beaver function is **uncomputable** and **asymptotically dominates** any computable function.
- **(PRA-)provably so,** no conjecture regarding the value of BB(1919) is (ZFC-)provable, not even "BB(1919) $= \heartsuit$" where $\heartsuit$ is the true value of BB(1919).

Part V

Very very very very large numbers

- **Rayo(n)** is the largest natural number uniquely definable using n symbols in the mathematical language of ZFC.
- The Rayo function is **(ZFC-)undefinable** and **asymptotically dominates** any (ZFC-)definable function.

award ceremony

86 submissions

category	number of submissions
disqualified	2
small-on-purpose	5
primes	2
nines	9
hyper	35
TREE	1
Busy Beaver	10
Rayo	2
referential	14
fun	9