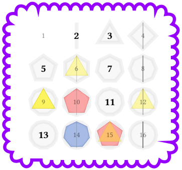


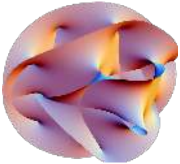

Exploring the internal language of toposes

– an invitation –

1 Alternate universes



2 Applications



3 Vision for the future

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6th World Congress and School on Universal Logic in Vichy
Workshop on Categories and Logic organized by Peter Arndt
June 22nd, 2018

A glimpse of the toposophic landscape

Set

The usual laws of logic hold.

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$\mathrm{Sh}\,X$

The intermediate value theorem fails.

Eff

Every function is computable.

A glimpse of the toposophic landscape

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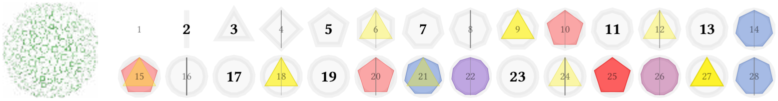

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The internal universe of a topos

A **topos** is a finitely complete cartesian closed category with a subobject classifier. For any topos $\mathcal{E}$ and any statement φ, we define the meaning of

$$\text{“}\mathcal{E} \models \varphi\text{”} \quad (\text{“}\varphi \text{ holds in the internal universe of } \mathcal{E}\text{”})$$

using the **Kripke–Joyal semantics**.

$\mathrm{Set} \models \varphi$	$\mathrm{Sh}(X) \models \varphi$	$\mathrm{Eff} \models \varphi$
“φ holds in the usual sense.”	“φ holds continuously.”	“φ holds computably.”

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“$\mathcal{E} \models \varphi$” (“$\varphi$ holds in the internal universe of $\mathcal{E}$”)

using the **Kripke–Joyal semantics**.

$\mathrm{Set} \models \varphi$ “φ holds in the usual sense.”	$\mathrm{Sh}(X) \models \varphi$ “φ holds continuously.”	$\mathrm{Eff} \models \varphi$ “φ holds computably.”

Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if $\varphi \vdash \psi$ intuitionistically, then $\mathcal{E} \models \psi$.

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Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if $\varphi \vdash \psi$ intuitionistically, then $\mathcal{E} \models \psi$.

no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

First steps in alternate universes

- Eff $\models$ "Any number is prime or is not prime." ✓
  Meaning: There is a **Turing machine** which determines of any given number whether it is prime or not.

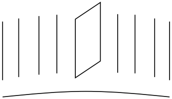
- Eff $\models$ "There are infinitely many prime numbers." ✓
  Meaning: There is a **Turing machine** producing arbitrarily many primes.

- Eff $\models$ "Any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not." ✗
  Meaning: There is a **Turing machine** which, given a Turing machine computing a function $f : \mathbb{N} \to \mathbb{N}$, determines whether f is zero or not.

- Eff $\models$ "Any function $\mathbb{N} \to \mathbb{N}$ is computable." ✓

- $\mathrm{Sh}(X) \models$ "Any cont. function with opposite signs has a zero." ✗
  Meaning: Zeros can locally be picked **continuously** in continuous families of continuous functions. (video for counterexample)

Applications



in commutative algebra

new reduction techniques
constructive proofs

in algebraic geometry

simpler and more conceptual proofs
synthetic algebraic geometry

in philosophy of math

historical contingency
platonism
pluralism
relativism

in further fields

synthetic measure theory
synthetic computability theory
synthetic homotopy theory (HoTT)
Bohr topos for quantum mechanics

The little Zariski topos of a ring

Let A be a reduced commutative ring ($x^n = 0 \Rightarrow x = 0$).

The **little Zariski topos** of A is equivalently

- the topos of sheaves over $\mathrm{Spec}(A)$,
- the classifying topos of local localizations of A or
- the classifying topos of prime filters of A

and contains a **mirror image** of A: $A^\sim$.

Assuming the Boolean prime ideal theorem, a first-order formula "$\forall \ldots \forall. (\cdots \Longrightarrow \cdots)$", where the two subformulas may not contain "$\Rightarrow$" and "$\forall$", holds for $A^\sim$ iff it holds for all stalks $A_\mathfrak{p}$.

$A^\sim$ inherits any property of A which is **localization-stable**.

$A^\sim$ is a **local ring** and a **field**.
$A^\sim$ has **$\neg\neg$-stable equality**.
$A^\sim$ is **anonymously Noetherian**.

The little Zariski topos of a ring

Let A be a reduced commutative ring ($x^n = 0 \Rightarrow x = 0$).

ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in $\mathbf{E}$ the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec($\mathbf{E}$, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

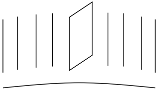
is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

Applications in commutative algebra

Let A be a reduced commutative ring ($x^n = 0 \Rightarrow x = 0$).
Let $A^\sim$ be its mirror image in the little Zariski topos.

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



A baby application

Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

Proof. M is also injective as a matrix over $A^\sim$. Since $A^\sim$ is a field, trivially $1 = 0$ in $A^\sim$. This means $1 = 0$ in A.

Generic freeness

Let M be a finitely generated A-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is a free $A[f^{-1}]$-module, then $1 = 0$ in A.

Proof. The claim is the translation of the statement "$M^\sim$ is **not not** free". Since $A^\sim$ is a field, this is trivial.

The Kripke–Joyal semantics for the little Zariski topos

Recall $A[f^{-1}] = \left\{ \frac{u}{f^n} \mid u \in A, n \in \mathbb{N} \right\}$.

- $\mathrm{Sh}(\mathrm{Spec}(A)) \models$ "For all $x \in A^\sim$, …"
  Meaning: For all $f \in A$ and all $x \in A[f^{-1}]$, …
- $\mathrm{Sh}(\mathrm{Spec}(A)) \models$ "There is $x \in A^\sim$ such that …"
  Meaning: There is a partition of unity, $1 = f_1 + \cdots + f_n \in A$, such that for each i, there exists $x_i \in A[f_i^{-1}]$ with …
- $\mathrm{Sh}(\mathrm{Spec}(A)) \models$ "φ implies ψ"
  Meaning: For all $f \in A$, if φ on stage f, then ψ on stage f.

Applications in algebraic geometry

Understand notions of algebraic geometry over a scheme X as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set
sheaf of modules	module
sheaf of finite type	finitely generated module
tensor product of sheaves	tensor product of modules
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$
dimension of X	Krull dimension of $\mathcal{O}_X$
spectrum of a sheaf of $\mathcal{O}_X$-algebras	ordinary spectrum [with a twist]
higher direct images	sheaf cohomology

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$\Longrightarrow$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  $$\text{set of prime ideals of } \mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n) + \text{Zariski topology} + \text{structure sheaf}$$

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

  $$A \longmapsto \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\}$$

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Synthetic approach: model schemes **directly as sets** in a certain nonclassical set theory, the internal universe of the **big Zariski topos** of a base scheme.

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

The big Zariski topos

The **big Zariski topos** $\mathrm{Zar}(S)$ of a scheme S is equivalently

1. the topos of sheaves over Aff/S,
2. the classifying topos of local rings over S or
3. the classifying $\mathrm{Sh}(S)$-topos of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.

- For an S-scheme X, its functor of points $\underline{X} = \mathrm{Hom}_S(\cdot, X)$ is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- Internally, $\underline{\mathbb{A}}^1$ (given by $\underline{\mathbb{A}}^1(X) = \mathcal{O}_X(X)$) looks like a field and is **synthetically quasicoherent**: For any finitely presented $\underline{\mathbb{A}}^1$-algebra B, the canonical map is bijective.

$$B \longrightarrow (\underline{\mathbb{A}}^1)^{\mathrm{Hom}_{\underline{\mathbb{A}}^1}(B, \underline{\mathbb{A}}^1)}, \; f \longmapsto (\varphi \mapsto \varphi(f))$$

- $\underline{\mathbb{P}}^n = [\![\{ (x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0 \} / (\underline{\mathbb{A}}^1)^\times]\!]$

Vision for the future

♥

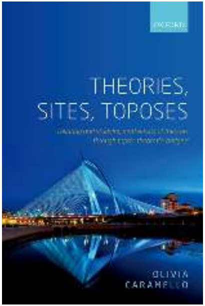
constructive algebra
conceptual proofs
proof assistants

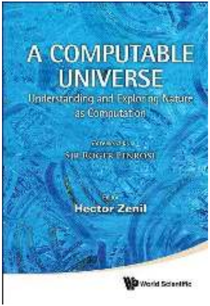
♥

constructive algebraic geometry
synthetic algebraic geometry
intersection theory
derived categories

♥

quasicoherence
nongeometric
mysteries
further fields






Spiel und Spaß mit der internen Welt des kleinen Zariski-Topos


Using the internal language of toposes in algebraic geometry