# How not to constructivize cohomology

*– interruptions welcome at any point –*

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104th Peripatetic Seminar on Sheaves and Logic in Amsterdam
October 6th, 2018

# Flabby sets

Let $M$ be a set. A subset $K \subseteq M$ is …

- a **subterminal** iff $\forall x, y \in K.\, x = y$.
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  iff $K \subseteq \{a\}$ for some $a \in M$.

Trivially, any subsingleton is a subterminal.

**Definition.** A set $M$ is **flabby** iff any subterminal is a subsingleton.

Any flabby set is inhabited.

Any flabby set is inhabited, for there is always the empty subterminal.

Conversely, given a set $M$ inhabited by some element $x_0 \in M$, it might appear that we have an easy proof that $M$ is flabby: Any subterminal $K \subseteq M$ is empty or of the form $K = \{x\}$ for some $x \in M$. In the first case, $K$ is a subsingleton for $K \subseteq \{x_0\}$, and in the second case, $K$ is a subsingleton for $K \subseteq \{x\}$.

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**Proof.** We have $M \hookrightarrow P(M)$, and $P(M)$ is flabby: Let $K \subseteq P(M)$ be a subterminal. Then $K \subseteq \{\bigcup K\}$, for if $A \in K$, then $K = \{A\}$ and hence $A \in \{\bigcup K\} = \{A\}$.

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**Open question.** Does any module embed into a flabby module?

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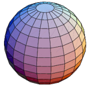
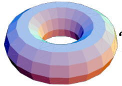
Even though constructively we can't show that any inhabited set is flabby, we can still verify that there are *enough flabby sets*.

However it's unknown whether there are *enough flabby modules*. (A module is *flabby* if and only if its underlying set is.) We'll see what the significance of this open question is later on.

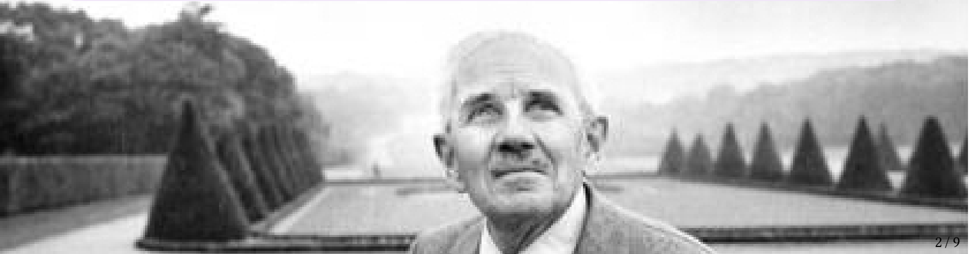
# Singular cohomology

Is  homeomorphic to ? No:

$H^0_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$ $H^1_{\text{sing}}($$, \mathbb{Z}) \cong 0$ $H^2_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$

$H^0_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$ $H^1_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ $H^2_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$

Given $f : X \to B$, can we compute the cohomology of $X$ if we understand the cohomology of $B$ and the cohomology of the fibers of $f$?





Associated to any topological space $X$ and any abelian group $A$ are the groups $H^n_{\text{sing}}(X, A)$, the singular cohomology groups of $X$ with coefficients in $A$. They depend functorially on $X$; hence one of many of their applications is to verify that given spaces are not homeomorphic.

Given a space $X$, we can hope that we can write $X$ as the codomain of a continuous map $f : X \to B$, in such a way that the base space $B$ and the fibers of $f$ are in some sense easy to understand. In such a situation we could ask whether the cohomology of $X$ can be computed from the cohomology of $B$ and the cohomology of the fibers.

The answer, given by Jean Leray in the 1940s, is: Yes, we can, but the framework of singular cohomology is too rigid for this task. For a positive answer we have to generalize to sheaf cohomology. And thus, the notion of sheaves was born.

# Sheaf cohomology

Let $E$ be a sheaf of modules over a space $X$. Let $\Gamma$ be the global sections functor. Choose an injective resolution $0 \to E \to I^0 \to I^1 \to \cdots$. Then the ***n*-th cohomology of *E* is**

$$\begin{aligned} H^n(X, E) &:= n\text{-th cohomology of } (0 \to \Gamma I^0 \to \Gamma I^1 \to \cdots) \\ &= \ker(\Gamma I^n \to \Gamma I^{n+1}) \,/\, \mathrm{im}(\Gamma I^{n-1} \to \Gamma I^n). \end{aligned}$$

- The modules $H^n(X, E)$ are important invariants. $[\ \chi(X, \mathcal{O}_X) = 1 - \mathrm{genus}_X,\ (C \cdot C') = \chi(\mathcal{O}_C \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}'_C), \ldots]$
- Let $A$ be an abelian group. Let $X$ be semi-locally contractible. Then $H^n(X, \underline{A}) = H^n_{\mathrm{sing}}(X, A)$ [Sella 2016].
- Let $f : X \to B$ be continuous. Then there is a spectral sequence $H^i(B, R^j f_*(E)) \Longrightarrow H^{i+j}(X, E)$.



An injective resolution is a sequence of sheaves of modules and linear morphisms as indicated such that the sequence is *exact* (the kernel of any outgoing morphism equals the image of the respective incoming morphism) and such that the sheaves $I^n$ are injective (a notion recalled below).

The fundamental fact of homological algebra is: Even though the sequence $0 \to I^0 \to I^1 \to \cdots$ can only fail to be exact at the front, the sequence $0 \to \Gamma I^0 \to \Gamma I^1 \to \cdots$ of global sections can fail to be exact at any place. Sheaf cohomology measures the extent of this failure.

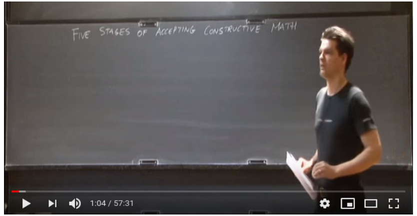
The standard proof of the existence of injective resolutions requires Zorn's lemma and the law of excluded middle. Injective resolutions are not unique, but the resulting sheaf cohomology modules are unique up to isomorphism. A primer on these matters is located here.

The positive answer to the question posed on the previous slide is given by the spectral sequence displayed at the bottom of this slide. The sheaves $R^n f_*(E)$ are called the *higher direct images of* $E$ (along $f$). Even if $E$ is a constant sheaf, its higher direct images might not be. This is the reason why singular cohomology is too restrictive.

The higher direct images $R^n f_*(E)$ are defined exactly as the sheaf cohomology $H^n(X, E)$, only with the global sections functor $\Gamma$ replaced by the pushforward functor $f_*$. They are dubbed "relative cohomology", for instance because under some conditions, there are isomorphisms $(R^n f_*(E))_b \cong H^n(X_b, E|_{X_b})$ where $X_b$ is the fiber of $b$ under $f$. This talk presents a rigorous and general way to regard higher direct images as sheaf cohomology.

# Constructive mathematics

mathematics without $\varphi \vee \neg\varphi$, $\neg\neg\varphi \Rightarrow \varphi$, axiom of choice




Andrej Bauer at an IAS talk

## Axiomatic freedom

"Every map $\mathbb{N} \to \mathbb{N}$ is computable."
"Every map $\mathbb{R} \to \mathbb{R}$ is continuous."
"Every map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is polynomial."
"Heyting Arithmetic has exactly one model."
"The subsets of $\{\heartsuit\}$ form a proper class."
"There is an injection $\mathbb{R} \to \mathbb{N}$."
⋮

## Applications

program extraction
synthetic differential geometry
synthetic algebraic geometry
synthetic domain theory
new reduction techniques in algebra
Bohr topos for quantum mechanics
⋮



Constructive mathematics can be studied for philosophical reasons or out of general mathematical curiosity. But restricting to constructive reasoning in our proofs also yields concrete gains for classical mathematics.

One of these is *program extraction*: From any constructive proof, we can mechanically extract a program witnessing the proven statement. A basic example is that any constructive proof of the infinitude of primes yields an algorithm for computing primes (together with a termination and correctness proof).

Another is that, since constructive mathematics is consistent with a number of anti-classical dream axioms, constructive mathematics allows to develop *synthetic accounts* of several subjects. For instance, in synthetic algebraic geometry, a scheme is just a set, a morphism of schemes is just a map of sets, and any map of the ground field into itself is polynomial.

There are also *reduction techniques* which propose interesting deals. For instance, there is a technique which allows us to pretend that a reduced ring is Noetherian and in fact a field – if in return we switch from classical reasoning to constructive. This particular technique has been used to turn the slightly convoluted multi-page proof of Grothendieck's generic freeness lemma into a simple one-paragraph argument.

An informative and entertaining primer on constructive mathematics can be found in the linked talk recording by Andrej Bauer or his written notes on the subject. Don't worry, the standard proof that $\sqrt{2}$ is not rational is perfectly fine in constructive mathematics.

# Relativization by internalization

Let $X$ be a space. The **internal language** of the topos $\mathrm{Sh}(X)$ allows us to reason about sheaves on $X$ in **naive element-based terms**.

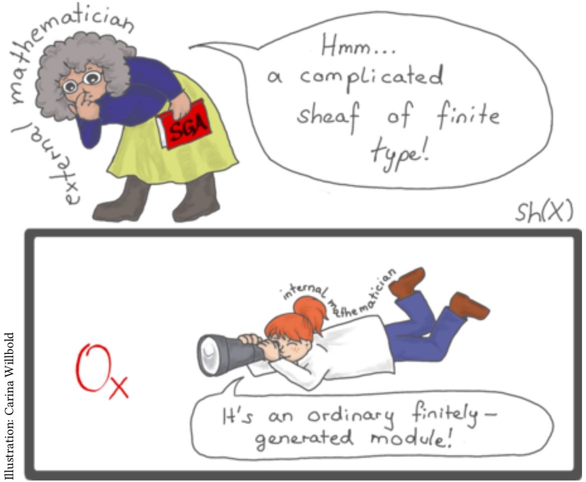



Illustration: Carina Willbold

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| sheaf | set/type |
| morphism of sheaves | map between sets |
| sheaf of cont. real-valued functions | set of Dedekind reals |
| over-locale $f : Y \to X$ | locale $I(Y)$ |
| sheaf over $Y$ | sheaf over $I(Y)$ |
| higher direct image $R^n f_* E$ | ?? sheaf cohomology $H^n(I(Y), E)$ |



The internal language of a topos $\mathcal{E}$ is a device which defines for any formula $\varphi$ of a certain language (a form of extensional type theory) what it means for $\varphi$ to hold *internally to* $\mathcal{E}$, written "$\mathcal{E} \models \varphi$". This translation process is sound with respect to intuitionistic logic; hence any theorem of constructive mathematics is valid in any topos. Only few toposes validate classical logic (for instance $\mathrm{Sh}(X)$ does if $X$ is a discrete space and the law of excluded middle is available in the metatheory).

As a special case, the internal language of the topos Set is just the usual mathematical language; more formally, Set $\models \varphi$ if and only if $\varphi$.

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| Every finite type sheaf of modules is finite locally free *on a dense open*. | Every finitely generated vector space is *not not* finite free. |
| In continuous families of continuous functions with opposite signs, zeros can locally be picked continuously. | The intermediate value theorem holds. |
| Grothendieck's generic freeness lemma holds. | (Some trivial observation about modules over fields.) |



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The intermediate value theorem ("any continuous function with opposite signs has a zero") doesn't admit a constructive proof, because for most spaces $X$ the external translation $\mathrm{Sh}(X) \models \mathrm{IVT}$ is not true – it's not true that in continuous families of continuous functions with opposite signs, zeros can locally be picked continuously, as this video shows.

Over reduced schemes, every finite type sheaf of modules is finite locally free on a dense open. This statement ("important hard exercise" 13.7.K in [Vakil 2017]) is just the external translation of the easy-to-prove internal statement that every finitely generated vector space does *not not* admit a finite basis. (A scheme is reduced if and only if its structure sheaf looks like a field from the internal point of view (in the sense that $1 \neq 0$ and $\neg(x \text{ invertible}) \Rightarrow x = 0$). This is why the reducedness condition is important.)

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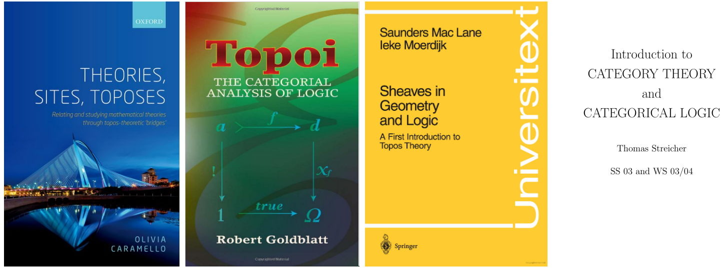
Excellent references on the internal language include:




We use an extension of the original form of the internal language which allows for unbounded quantification, Mike Shulman's *stack semantics*. (Independently, Steve Awodey, Carsten Butz, Alex Simpson and Thomas Streicher developed a similar semantics.)

# Internalizing higher direct images

A set $M$ is **injective** iff for any injection $A \to B$, any map $A \to M$ extends to a map on $B$.

- "A set is injective iff it's inhabited" is a **constructive taboo**.
- Constructively, there are still **enough injective sets**.
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A module $M$ is **injective** iff for any linear injection $A \to B$, any linear map $A \to M$ extends to a linear map on $B$.

- It's consistent with **ZF** that there are no injective modules [Blass 1979].
- The existence of enough injective modules is **constructively neutral**.

Somewhat surprisingly, even though the standard proof that there are enough injective modules requires the axiom of choice and even though it's consistent with Zermelo–Fraenkel set theory that the zero module is the only injective $\mathbb{Z}$-module, the existence of enough injective modules is *constructively neutral*, that is, does not imply a fundamentally nonconstructive principle like the law of excluded middle.

Indeed, assuming the axiom of choice in the metatheory, the statement "any module can be embedded into an injective module" holds in the internal language of any Grothendieck topos. This is because, assuming the axiom of choice in the metatheory, any sheaf of modules over a site can be embedded into an injective sheaf of modules and, somewhat surprisingly, …

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A sheaf of modules $M$ is **injective** iff for any linear monomorphism $A \rightarrowtail B$, any linear morphism $A \to M$ extends to a linear morphism on $B$.

- Assuming choice, there are enough injectives over any site.
- Assuming Zorn's lemma, a sheaf of modules over a locale $X$ is injective iff, from the internal point of view of $\mathrm{Sh}(X)$, it is an injective module.



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… a sheaf of modules is injective if and only if it is an injective module from the internal point of view. The "$\Rightarrow$" direction is straightforward; the "$\Leftarrow$" direction is nontrivial: The external meaning of the internal existential quantifier is *local* existence. Hence linear morphisms into a sheaf of modules which is injective from the internal point of view can *locally* be extended. But these extensions need not be compatible, hence might not glue to a global extension. For the case of sheaves of abelian groups, this result is due to Roswitha Harting in an 1983 paper of her. The case of sheaves of modules is arguably also due to her, even though she states that the result doesn't hold for sheaves of modules. (Technology has improved since then, and using flabbiness as an organizing principle one can give a reasonably straightforward proof of the general statement.)

In contrast, internally and externally projective modules do not coincide at all.

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A consequence of the fact that internal and external injectivity coincides for sheaves of modules over locales is that we can interpret the higher direct images $R^n f_*(Y, E)$ of a sheaf $E$ of modules over an over-locale $f : Y \to X$ as sheaf cohomology $H^n(I(Y), E)$, where $I(Y)$ is the internal locale of $\mathrm{Sh}(X)$ corresponding to $Y$.

A basic application is the following. Any student in algebraic geometry needs, at some point in her life, to compute the cohomology of projective space. At a later point she needs to compute higher direct images along $\mathbb{P}^n_S \to S$, where $\mathbb{P}^n_S$ is a relative version of projective space. Since higher direct images are just internal sheaf cohomology, she can in fact skip the second computation.

Further progress along these lines is hindered by the fact that we don't yet have a constructive account of sheaf cohomology.

# Flabby resolutions

A sheaf $E$ on a space $X$ is **flabby** iff any local section $s \in E(U)$ on an open $U$ extends to a global section $\bar{s} \in E(X)$: $\bar{s}|_U = s$.

- Assuming Zorn's lemma:
  A sheaf is flabby iff, from the internal point of view, it's a flabby set.
- Assuming the law of excluded middle:
  Any sheaf of modules over a topological space embeds into a flabby sheaf of modules.
- Assuming Zorn's lemma, flabby sheaves of modules are **acyclic for the global sections functor**. Hence, assuming ??, sheaf cohomology and higher direct images can be computed using **flabby resolutions**.

Since we cannot show the existence of enough injective sheaves of modules (or even just plain modules) constructively, the definition of sheaf cohomology using injective resolutions doesn't work in a constructive setting. Classically it's known that *flabby resolutions* can also be used to compute sheaf cohomology. There are more flabby sheaves than injective ones, they have better stability properties (flabby sheaves are preserved under pushforward) and the axiom of choice is not needed to construct flabby resolutions (the standard proof uses only the law of excluded middle, and not Zorn's lemma). Hence it seems reasonable to base a constructively sensible definition of sheaf cohomology on flabby resolutions. We tried to do so, and failed.

Assuming Zorn's lemma, the notion of a flabby sheaf is a local notion, meaning that a sheaf is flabby if and only if its restrictions to every member of an open covering are, but this fact is not obvious from the definition. In contrast, the notion that a sheaf $E$ is flabby from the internal point of view is local without any assumptions (as is any internal notion), hence maybe we should consider adopting internal flabbiness as the official definition of flabbiness. Its external translation is:

A sheaf $E$ is flabby from the internal point of view if and only if for any local section $s \in E(U)$, there is an open covering $X = \bigcup_i U_i$ such that for all $i$, the section $s$ extends to a section on $U \cup U_i$.

# Flabbiness as an organizing principle

**Proposition.** Let $M$ be a sheaf of modules over a locale $X$. Then $M$ is injective iff it is injective from the point of view of $\mathrm{Sh}(X)$.

**Proof.** (Only "$\Leftarrow$".) Let $i : A \to B$ be a linear monomorphism. Let $f : A \to M$ be a linear morphism. We verify, internally, that the set $E := \{\bar{f} : B \to M \mid \bar{f} \circ i = f\}$ is flabby.

The notion of being flabby from the internal point of view turns out to have valuable organizing power. For instance, both of the following statements can be proven by first verifying that a certain sheaf is internally flabby (which can be done entirely constructively) and then appealing to Zorn's lemma in order to obtain a global section of that sheaf.

- Flabby sheaves are acyclic for the global sections functor: Let $0 \to E \to F \to G \to 0$ be a short exact sequence of sheaves of modules and let $E$ be flabby. Then the sequence remains exact after taking global sections.

  (Verify that the sheaf of local preimages of a given section $s \in G(X)$ is flabby.)
- Internally injective modules are externally injective.

  (See proof on the slide.)

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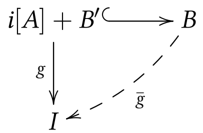
Let $K \subseteq E$ be a subterminal. We consider the injectivity diagram




where $B' := \{t \in B \mid t = 0 \text{ or } K \text{ is inhabited}\} \subseteq B$ and $g$ is defined as follows: Let $s \in i[A] + B'$. Then $s = i(a) + t$ for some $a \in A$ and $t \in B'$. Since $t \in B'$, $t = 0$ or $K$ is inhabited. If $t = 0$, we set $g(s) := f(a)$. If $K$ is inhabited, we set $g(s) := f(a) + \bar{f}(s)$, where $\bar{f}$ is any element of $K$.

Since $M$ is injective, there exists a dotted map $\bar{g} \in E$. We have $K \subseteq \{\bar{g}\}$. $\square$



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# Flabbiness in the effective topos

A set $M$ is **flabby** iff any subterminal $K \subseteq M$ is a subsingleton.

$\forall x, y \in K.\, x = y$ $\qquad$ $\exists a \in M.\, K \subseteq \{a\}$

**Proposition.** Let $X$ be an effective object in the effective topos. Then

"If $X$ is flabby, any endomap on $X$ has a fixed point."

from the point of view of the effective topos.

**Proof (sketch).** We have a procedure which computes for any subterminal $K \subseteq X$ an element $a_K$ such that $K \subseteq \{a_K\}$. Let $f : X \to X$ be a map. Construct $K := \{f(a_K)\}$. Then $K \subseteq \{a_K\}$, so $f(a_K) = a_K$. $\square$

**Corollary.** The only effective flabby module $M$ is the zero module.

**Proof.** Let $x \in M$. Then $x + a = a$ for some $a \in M$; hence $x = 0$. $\square$

**Proposition.** Assuming the law of excluded middle, any $\neg\neg$-separated module in the effective topos can be embedded into a flabby module.

**Proof.** We have $M \hookrightarrow \Delta\Gamma M$. $\square$

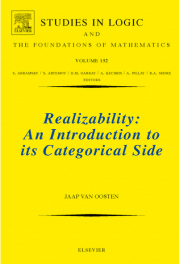


The notion of flabby sets was conceived to model the notion of flabby sheaves and is therefore closely connected to Grothendieck toposes. Hence it is instructive to study flabby objects in elementary toposes which are not Grothendieck toposes, away from their original conceptual home.

In particular, we hope to prove the conjecture that the statement "any module embeds into a flabby module" is not constructively provable by verifying that it doesn't hold in the effective topos.

To this end, the slide displays two results.

Details on the self-referential construction "$K := \{f(a_K)\}$" are in this draft paper. A module is $\neg\neg$-separated if and only if $\neg\neg(x = 0)$ implies $x = 0$. References on the effective topos include Martin Hyland's survey paper and the canonical book by one of our honoraries:



# State of affairs

☺

The existence of enough injective modules is constructively neutral.
Higher direct images can be understood as internal sheaf cohomology.

☹

Flabby sheaves can fail to be acyclic, constructively.
There is still no general constructive framework for sheaf cohomology.
Even though:

- Basic homological algebra is entirely constructive.
- There are algorithms for computing cohomology [Barakat, …].
- Čech methods work constructively, even in a synthetic context.



Even if we could constructively prove that there are enough flabby modules, there is still the problem that the proof that flabby sheaves of modules are acyclic for the global sections functor (appears to) require Zorn's lemma.

Hence it appears that the simple idea of basing a constructive account of sheaf cohomology on flabby resolutions doesn't work.

More work is needed. I hope that some day, we can study the cohomology of the smallest dense sublocale of the one-point space.*

* Assuming the law of excluded middle, this locale is just the one-point space again. Hence cohomology of this locale should measure the extent to which we're nonclassical, being zero if and only if the law of excluded middle holds.

An alternative way of putting this question is as follows. Let $\mathrm{Set}_{\neg\neg}$ be the smallest dense subtopos of Set, the topos of double negation sheaves. The forgetful functor $\mathrm{Ab}(\mathrm{Set}_{\neg\neg}) \to \mathrm{Ab}$ is left-exact, but might not be right-exact, since a map $f : A \to B$ is an epimorphism in $\mathrm{Ab}(\mathrm{Set}_{\neg\neg})$ if and only if $\forall y \in B.\, \neg\neg(\exists x \in A.\, f(x) = y)$, which is weaker than being surjective. What do its right derived functors look like?