Bridging the foundational gap:

Updating **algebraic geometry** in face of **current challenges** regarding **formalizability**, **constructivity** and **predicativity**

– an invitation –

Dagstuhl Seminar 20202:
Geometric Logic, Constructivisation, and Automated Theorem Proving

November 23th, 2021

Ingo Blechschmidt
University of Augsburg

Algebraic geometry in a nutshell

Turn **commutative rings** into **spaces**, and **glue** those spaces.

Examples

1. $\mathrm{Spec}(k[X_1, \ldots, X_n]) = \mathbb{A}^n$.
2. $\mathrm{Spec}(k[X, Y]/(Y - X^2))$ = standard parabola.
3. Gluing  with  along  yields $\mathbb{P}^1$.

Concrete results

Fermat's Last Theorem: For $n \geq 3$, no positive integers satisfy

$$a^n + b^n = c^n.$$

Transfinite methods

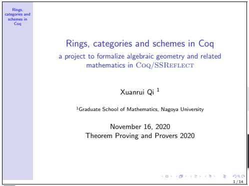
The standard presentation of algebraic geometry hinges on:

- large structures: classes, large categories, universes, …
- powersets
- law of excluded middle
- axiom of choice

despite:

1. subject matter (in part) very concrete
2. computer algebra systems for computations practical
3. constructive algebra well-established
4. high-level proofs often constructive

Formalizing algebraic geometry



Rings, categories and schemes in Coq

a project to formalize algebraic geometry and related mathematics in Coq/SSReflect

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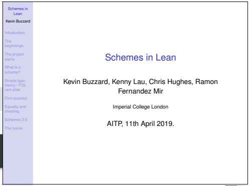
November 16, 2020
Theorem Proving and Provers 2020

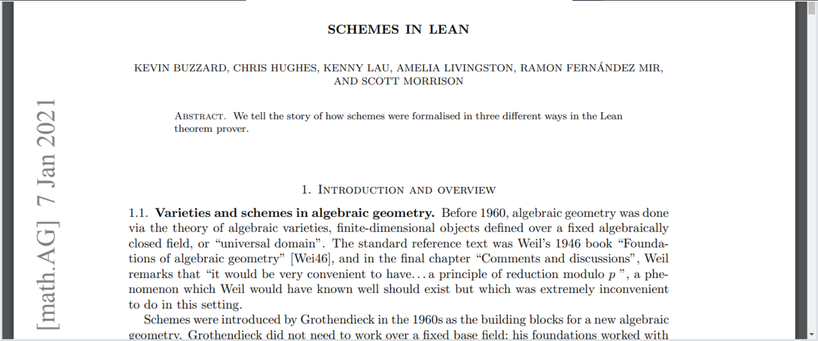


Schemes in Lean

Kevin Buzzard, Kenny Lau, Chris Hughes, Ramon Fernandez Mir

Imperial College London

AITP, 11th April 2019.



SCHEMES IN LEAN

KEVIN BUZZARD, CHRIS HUGHES, KENNY LAU, AMELIA LIVINGSTON, RAMON FERNÁNDEZ MIR, AND SCOTT MORRISON

ABSTRACT. We tell the story of how schemes were formalised in three different ways in the Lean theorem prover.

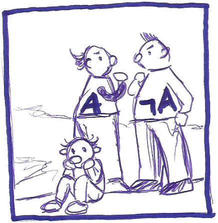
1. INTRODUCTION AND OVERVIEW

1.1. **Varieties and schemes in algebraic geometry.** Before 1960, algebraic geometry was done via the theory of algebraic varieties, finite-dimensional objects defined over a fixed algebraically closed field, or "universal domain". The standard reference text was Weil's 1946 book "Foundations of algebraic geometry" [Wei46], and in the final chapter "Comments and discussions", Weil remarks that "it would be very convenient to have... a principle of reduction modulo p", a phenomenon which Weil would have known well should exist but which was extremely inconvenient to do in this setting.

Schemes were introduced by Grothendieck in the 1960s as the building blocks for a new algebraic geometry. Grothendieck did not need to work over a fixed base field; his foundations worked with

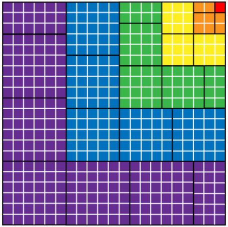
[math.AG] 7 Jan 2021

A trinitarian challenge



constructivity

$$1^3 + \cdots + n^3 = (1 + \cdots + n)^2$$



elegance

formalizability

Thesis

Elegant mathematics lends itself to rewarding formalization.

Foundational possibilities

Schemes and scheme morphisms

- locally ringed …
  - … topological spaces – *unconstructive, hard to formalize*
  - … locales – *impredicative, superfluous opens*
  - … distributive lattices – *need extension from basis*
  - … sites – *current favorite! morphisms intricate*
  - … toposes – *large structure*
  - … arithmetic universes – *a bit better; issue with relative spectrum*
- formal geometries – *can be regarded as sites*
- functor of points – *large structure, issue with sizes or with schemes not of finite presentation over the base ring*
- formal gluing data – *morphisms intricate*

A surprise of uncertain import:

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A surprise of uncertain import: Internally to the big Zariski topos of a base scheme, the Zariski spectrum of a finitely presented algebra does have enough points! [Cherubini–Coquand]

Foundational possibilities

Schemes and scheme morphisms

Zariski cohomology

- Čech cohomology – *ad hoc, but fine for quasicompact separated schemes*
- injective resolutions – *hopelessly unconstructive*
- dynamical injectives – *??*
- flabby resolutions – *probably unconstructive*
- pointwise Kan extensions – *fine! partially defined* $\mathbb{R}\Gamma$*;*
  – *existence verified for quasicompact separated schemes; hyper coverings?*

Étale cohomology

??