How topos theory can help commutative algebra

– an invitation –

1 **Background on the internal language**

2 **Applications in commutative algebra**

3 **The mystery of nongeometric sequents**

Ingo Blechschmidt (MPI Leipzig)
Toposes in Como
June 29th, 2018

Motivating test cases

Let A be a ring (commutative, with unit, $1 = 0$ allowed).
Assume that A is reduced: If $x^n = 0$, then $x = 0$.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

A baby application

Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

A child application

Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

The two displayed statements are trivial for fields. It is therefore natural to try to reduce the general situation to the field situation.

Motivating test cases

Let A be a ring (commutative, with unit, $1 = 0$ allowed).
Assume that A is reduced: If $x^n = 0$, then $x = 0$.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

A baby application

Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

Proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over the field $A/\mathfrak{m}$. This is a contradiction to basic linear algebra.

A child application

Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

Proof. Assume not. Then there is a minimal prime ideal $\mathfrak{p}$. The matrix is injective over the field $A_\mathfrak{p} = A[(A \setminus \mathfrak{p})^{-1}]$. This is a contradiction to basic linear algebra.

The two displayed statements are trivial for fields. It is therefore natural to try to reduce the general situation to the field situation.

The displayed proofs, which could have been taken from any standard textbook on commutative algebra, succeed in this reduction quite easily by employing maximal ideals or minimal prime ideals. However, this way of reducing comes at a cost: It requires the Boolean Prime Ideal Theorem (for ensuring the existence of a prime ideal and for ensuring that stalks at minimal prime ideals are fields) and even the full axiom of choice (for ensuring the existence of a minimal prime ideal).

It therefore doesn't work in the internal universe of most toposes, and in any case it obscures explicit computational content: Statements so simple as the two displayed ones should admit explicit, computational proofs.

We'll learn how the internal language of a certain well-chosen topos provides a way to perform the reduction in an entirely constructive manner. If so desired, the resulting topos-theoretic proofs can be unwound to yield fully explicit, topos-free, direct proofs.

Beautiful constructive proofs can also be found in Richman's note on nontrivial uses of trivial rings and in the recent textbook by Lombardi and Quitté on constructive commutative algebra.

Motivating test cases

Let A be a ring (commutative, with unit, $1 = 0$ allowed).
Assume that A is reduced: If $x^n = 0$, then $x = 0$.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

A baby application

Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

Proof. Assume not. Then there is a prime ideal $\mathfrak{p}$. The matrix is surjective over the field $\mathrm{Quot}(A/\mathfrak{p})$. This is a contradiction to basic linear algebra.

A child application

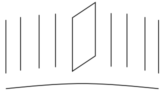
Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

Proof. Assume not. Then there is a minimal prime ideal $\mathfrak{p}$. The matrix is injective over the field $A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$. This is a contradiction to basic linear algebra.

We can slightly reduce the requirements of the proof of the first statement by employing not a maximal ideal, but a prime ideal. The existence of maximal ideals in nontrivial rings is equivalent to the axiom of choice, while the existence of prime ideals is equivalent to the weaker Boolean Prime Ideal Theorem. However, this improvement doesn't change the main point; the proof is still wildly nonconstructive.

Motivating test cases

Let A be a ring (commutative, with unit, $1 = 0$ allowed).
Assume that A is reduced: If $x^n = 0$, then $x = 0$.



Generic freeness

Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
Let M be a finitely generated B-module ($\cong B^m/U$).
If $f = 0$ is the only element of A such that

1. $B[f^{-1}]$ and $M[f^{-1}]$ are free modules over $A[f^{-1}]$,
2. $A[f^{-1}] \to B[f^{-1}]$ is of finite presentation and
3. $M[f^{-1}]$ is finitely presented as a module over $B[f^{-1}]$,

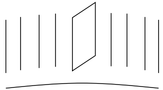
then $1 = 0$ in A.

Grothendieck's generic freeness lemma is an important theorem in algebraic geometry, where it is usually stated in the following geometric form:

> Let X be a reduced scheme. Let $\mathcal{B}$ be an $\mathcal{O}_X$-algebra of finite type. Let $\mathcal{M}$ be a $\mathcal{B}$-module of finite type. Then over a dense open,
>
> (a) $\mathcal{B}$ and $\mathcal{M}$ are locally free as sheaves of $\mathcal{O}_X$-modules,
> (b) $\mathcal{B}$ is of finite presentation as a sheaf of $\mathcal{O}_X$-algebras and
> (c) $\mathcal{M}$ is of finite presentation as a sheaf of $\mathcal{B}$-modules.

Motivating test cases

Let A be a ring (commutative, with unit, $1 = 0$ allowed).
Assume that A is reduced: If $x^n = 0$, then $x = 0$.



Generic freeness

Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
Let M be a finitely generated B-module ($\cong B^m/U$).
If $f = 0$ is the only element of A such that

1. $B[f^{-1}]$ and $M[f^{-1}]$ are free modules over $A[f^{-1}]$,
2. $A[f^{-1}] \to B[f^{-1}]$ is of finite presentation and
3. $M[f^{-1}]$ is finitely presented as a module over $B[f^{-1}]$,

then $1 = 0$ in A.

Proof. See [Stacks Project, Tag 051Q].

Grothendieck's generic freeness lemma is an important theorem in algebraic geometry, where it is usually stated in the following geometric form:

> Let X be a reduced scheme. Let $\mathcal{B}$ be an $\mathcal{O}_X$-algebra of finite type. Let $\mathcal{M}$ be a $\mathcal{B}$-module of finite type. Then over a dense open,
>
> (a) $\mathcal{B}$ and $\mathcal{M}$ are locally free as sheaves of $\mathcal{O}_X$-modules,
> (b) $\mathcal{B}$ is of finite presentation as a sheaf of $\mathcal{O}_X$-algebras and
> (c) $\mathcal{M}$ is of finite presentation as a sheaf of $\mathcal{B}$-modules.

All previously known proofs proceed in a series of reduction steps, finally culminating in the case where A is a Noetherian integral domain. They are somewhat convoluted (for instance, the proof in the Stacks Project is three pages long) and employ several results in commutative algebra which have not yet been constructivized.

Using the internal language of toposes, we will give a short, conceptual and constructive proof of Grothendieck's generic freeness lemma. Again, if so desired, one can unwind the internal proof to obtain a constructive proof which doesn't reference topos theory. The proof obtained in this way is still an improvement on the previously known proofs, requiring no advanced prerequisites in commutative algebra, and takes about a page.

The internal language of a topos

For any topos $\mathcal{E}$ and any formula φ, we define the meaning of

"$\mathcal{E} \models \varphi$" ("$\varphi$ holds in the internal universe of $\mathcal{E}$")

using (Shulman's extension of) the **Kripke–Joyal semantics**.

$\mathrm{Set} \models \varphi$	$\mathrm{Sh}(X) \models \varphi$	$\mathrm{Eff} \models \varphi$
"φ holds in the usual sense."	"φ holds continuously."	"φ holds computably."

Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if φ entails ψ intuitionistically, then $\mathcal{E} \models \psi$.

The internal language of a topos allows to construct objects and morphisms of the topos, formulate statements about them and prove such statements *in a naive element-based language*. From the internal point of view, objects look like sets [more precisely, types]; morphisms look like maps; monomorphisms look like injections; epimorphisms look like surjections; group objects look like groups; and so on.

To determine whether a statement φ holds in the internal universe of a given topos, we can use the Kripke–Joyal semantics to translate it into an ordinary external statement and then check the validity of the external translation.

For instance, in the effective topos the curious statement "any function $\mathbb{N} \to \mathbb{N}$ is computable" holds, for its external meaning is the triviality "there is a Turing machine which given a Turing machine computing some function $f : \mathbb{N} \to \mathbb{N}$ outputs a Turing machine computing f". In contrast, the statement "any function $\mathbb{N} \to \mathbb{N}$ is either the zero function or not" does not hold in the effective topos, since its external meaning is "there exists a Turing machine which given a Turing machine computing some function $f : \mathbb{N} \to \mathbb{N}$ decides whether f is the zero function or not".

The internal language of a topos

For any topos $\mathcal{E}$ and any formula φ, we define the meaning of

"$\mathcal{E} \models \varphi$" ("$\varphi$ holds in the internal universe of $\mathcal{E}$")

using (Shulman's extension of) the **Kripke–Joyal semantics**.

$\mathrm{Set} \models \varphi$	$\mathrm{Sh}(X) \models \varphi$	$\mathrm{Eff} \models \varphi$
"φ holds in the usual sense."	"φ holds continuously."	"φ holds computably."

Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if φ entails ψ intuitionistically, then $\mathcal{E} \models \psi$.

no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

Any theorem which has an intuitionistic proof holds in the internal universe of any topos. The restriction to intuitionistic logic is not due to philosophical concerns; it is a fact of life that only very few toposes validate the law of excluded middle (for instance, sheaf toposes over discrete topological spaces do if the law of excluded middle is available in the metatheory). Luckily, vast amounts of mathematics can be developed in a purely intuitionistic setting.

The internal language machinery itself can be developed in an intuitionistic setting.

The standard internal language of toposes in not enough for our purposes, as it misses unbounded quantification ("for all groups", "for all rings") and dependent types. Shulman's stack semantics offers what we need. No knowledge of stacks is necessary to enjoy his paper. Prior work includes Polymorphism is Set Theoretic, Constructively by Pitts and Relating first-order set theories, toposes and categories of classes by Awodey, Butz, Simpson and Streicher (obtained independently and published after Shulman).

The Kripke–Joyal semantics of $\mathrm{Sh}(X)$

Let X be a topological space. We recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ. Write “$\mathrm{Sh}(X) \models \varphi$” to mean $X \models \varphi$.

$U \models \top$	iff	true
$U \models \bot$	iff	~~false~~ $U = \emptyset$
$U \models s = t : F$	iff	$s\vert_U = t\vert_U \in F(U)$
$U \models \varphi \wedge \psi$	iff	$U \models \varphi$ and $U \models \psi$
$U \models \varphi \vee \psi$	iff	~~$U \models \varphi$ or $U \models \psi$~~ there exists a covering $U = \bigcup_i U_i$ such that for all i: $U_i \models \varphi$ or $U_i \models \psi$
$U \models \varphi \Rightarrow \psi$	iff	for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$
$U \models \forall s : F.\, \varphi(s)$	iff	for all open $V \subseteq U$ and sections $s_0 \in F(V)$: $V \models \varphi(s_0)$
$U \models \forall F.\, \varphi(F)$	iff	for all open $V \subseteq U$ and sheaves F_0 over V: $V \models \varphi(F_0)$
$U \models \exists s : F.\, \varphi(s)$	iff	~~there exists $s_0 \in F(U)$ such that $U \models \varphi(s_0)$~~ there exists a covering $U = \bigcup_i U_i$ such that for all i: there exists $s_0 \in F(U_i)$ such that $U_i \models \varphi(s_0)$
$U \models \exists F.\, \varphi(F)$	iff	~~there exists a sheaf F_0 on U such that $U \models \varphi(F_0)$~~ there exists a covering $U = \bigcup_i U_i$ such that for all i: there exists a sheaf F_0 on U_i such that $U_i \models \varphi(F_0)$

Many interesting sheaves have few global sections, which is why a definition such as “$U \models \forall s : F.\, \varphi(s)$ iff $U \models \varphi(s_0)$ for all $s_0 \in F(U)$” would miss the point.

Here is an explicit example of the translation procedure. Let $\alpha : F \to G$ be a morphism of sheaves on X. Then (the corner quotes “$\ulcorner \ldots \urcorner$” indicate that translation into formal language is left to the reader):

$X \models \ulcorner \alpha \text{ is injective} \urcorner$

$\iff X \models \forall s : F.\, \forall t : F.\, \alpha(s) = \alpha(t) \Rightarrow s = t$

$\iff$ for all open $U \subseteq X$, sections $s_0 \in F(U)$:
$U \models \forall t : F.\, \alpha(s_0) = \alpha(t) \Rightarrow s_0 = t$

$\iff$ for all open $U \subseteq X$, sections $s_0 \in F(U)$:
for all open $V \subseteq U$, sections $t_0 \in F(V)$:
$V \models \alpha(s_0) = \alpha(t_0) \Rightarrow s_0 = t_0$

$\iff$ for all open $U \subseteq X$, sections $s_0 \in F(U)$:
for all open $V \subseteq U$, sections $t_0 \in F(V)$:
for all open $W \subseteq V$: $\alpha_V(s_0|_W) = \alpha_V(t_0|_W)$ implies $s_0|_W = t_0|_W$

$\iff$ for all open $U \subseteq X$, sections $s, t \in F(U)$: $\alpha_U(s) = \alpha_U(t)$ implies $s = t$

$\iff \alpha$ is a monomorphism of sheaves

Internalizing parameter-dependence

Let X be a space. A continuous family $(f_x)_{x \in X}$ of continuous functions (that is, a continuous function $f : X \times \mathbb{R} \to \mathbb{R}; f_x(a) = f(x, a)$) induces an endomorphism of the sheaf $\mathcal{C}$ of continuous functions:

$$\bar{f} : \mathcal{C} \longrightarrow \mathcal{C}, \text{ on } U\text{: } s \longmapsto (x \mapsto f_x(s(x))).$$

- $\mathrm{Sh}(X) \models \ulcorner$The set $\mathcal{C}$ is the set of (Dedekind) reals$\urcorner$.
- $\mathrm{Sh}(X) \models \ulcorner$The function $\bar{f} : \mathbb{R} \to \mathbb{R}$ is continuous$\urcorner$.

- Iff $f_x(-1) < 0$ for all x, then $\mathrm{Sh}(X) \models \bar{f}(-1) < 0$.
- Iff $f_x(+1) > 0$ for all x, then $\mathrm{Sh}(X) \models \bar{f}(+1) > 0$.
- Iff all f_x are increasing, then $\mathrm{Sh}(X) \models \ulcorner\bar{f}$ is increasing$\urcorner$.

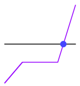
- Iff there is an open cover $X = \bigcup_i U_i$ such that for each i there is a continuous function $s : U_i \to \mathbb{R}$ with $f_x(s(x)) = 0$ for all $x \in U_i$, then $\mathrm{Sh}(X) \models \exists s\,{:}\,\mathbb{R}.\, \bar{f}(s) = 0$.

This slide, unrelated to commutative algebra or algebraic geometry, aims to illustrate one of the basic uses of the internal language of toposes: Upgrading any theorem admitting an intuitionistic proof to a parameter-dependent version.

Constructively, there are several non-equivalent forms of the intermediate value theorem. The following version doesn't admit an intuitionistic proof:

> Let $g : \mathbb{R} \to \mathbb{R}$ be a function between the (Dedekind) reals which is continuous in the usual epsilon/delta sense. Assume $g(-1) < 0 < g(1)$. Then there exists a number $x \in \mathbb{R}$ such that $g(x) = 0$.

If there was an intuitionistic proof, the statement would hold in any topos, so in particular in sheaf toposes over topological spaces. By the translations shown on the slide, this would amount to the following strengthening of the intermediate value theorem: In continuous families of continuous functions, zeros can locally be picked continuously. However, this strengthening is invalid, as this video shows.



Internalizing parameter-dependence

Let X be a space. A continuous family $(f_x)_{x \in X}$ of continuous functions (that is, a continuous function $f : X \times \mathbb{R} \to \mathbb{R}$; $f_x(a) = f(x, a)$) induces an endomorphism of the sheaf $\mathcal{C}$ of continuous functions:

$$\bar{f} : \mathcal{C} \longrightarrow \mathcal{C}, \text{ on } U\text{: } s \longmapsto (x \mapsto f_x(s(x))).$$

- $\mathrm{Sh}(X) \models \ulcorner$The set $\mathcal{C}$ is the set of (Dedekind) reals$\urcorner$.
- $\mathrm{Sh}(X) \models \ulcorner$The function $\bar{f} : \mathbb{R} \to \mathbb{R}$ is continuous$\urcorner$.

- Iff $f_x(-1) < 0$ for all x, then $\mathrm{Sh}(X) \models \bar{f}(-1) < 0$.
- Iff $f_x(+1) > 0$ for all x, then $\mathrm{Sh}(X) \models \bar{f}(+1) > 0$.
- Iff all f_x are increasing, then $\mathrm{Sh}(X) \models \ulcorner \bar{f}$ is increasing$\urcorner$.

- Iff there is an open cover $X = \bigcup_i U_i$ such that for each i there is a continuous function $s : U_i \to \mathbb{R}$ with $f_x(s(x)) = 0$ for all $x \in U_i$, then $\mathrm{Sh}(X) \models \exists s : \mathbb{R}.\, \bar{f}(s) = 0$.

In contrast, the following version does admit an intuitionistic proof. We can therefore interpret it in sheaf toposes over topological spaces and thereby obtain the strengthening that in continuous families of strictly increasing continuous functions, zeros can locally be picked continuously. You are invited to prove this strengthening directly, without reference to the internal language.

> Let $g : \mathbb{R} \to \mathbb{R}$ be a function between the (Dedekind) reals which is continuous in the usual epsilon/delta sense and which is strictly increasing ($a < b$ implies $g(a) < g(b)$). Assume $g(-1) < 0 < g(1)$. Then there exists a number $x \in \mathbb{R}$ such that $g(x) = 0$.

The little Zariski topos

Let A be a ring. Its **little Zariski topos** is equivalently

1. the classifying topos of **local localizations** of A,
2. the classifying locale of **prime filters** of A,
3. the locale given by the frame of **radical ideals** of A,
4. the topos of sheaves over the poset A with $f \preceq g$ iff $f \in \sqrt{(g)}$ and with $(f_i \to f)_i$ deemed covering iff $f \in \sqrt{(f_i)_i}$ or
5. the topos of sheaves over $\operatorname{Spec}(A)$.

Its associated topological space of points is the **classical spectrum**

$$\{\mathfrak{f} \subseteq A \mid \mathfrak{f} \text{ prime filter}\} + \text{Zariski topology}.$$

It has **enough points** if the Boolean Prime Ideal Theorem holds.

Prime ideal: $0 \in \mathfrak{p};\ x \in \mathfrak{p} \wedge y \in \mathfrak{p} \Rightarrow x + y \in \mathfrak{p};\ 1 \notin \mathfrak{p};\ xy \in \mathfrak{p} \Leftrightarrow x \in \mathfrak{p} \vee y \in \mathfrak{p}$

Prime filter: $0 \notin \mathfrak{f};\ x + y \in \mathfrak{f} \Rightarrow x \in \mathfrak{f} \vee y \in \mathfrak{f};\ 1 \in \mathfrak{f};\ xy \in \mathfrak{f} \Leftrightarrow x \in \mathfrak{f} \wedge y \in \mathfrak{f}$

Any geometric theory has a classifying topos; if the theory under consideration is propositional (doesn't have any sorts), then its classifying topos can be chosen to be the topos of sheaves over a locale. One can also give a direct account of classifying locales, as a pedagogical stepping stone to the full theory of classifying toposes.

The slide contains a small lie: The classical definition of the spectrum of a ring is via the set of prime ideals of A, not prime filters. If the law of excluded middle is available, there is no difference between these definitions since the complement of a prime ideal is a prime filter and vice versa.

One can also consider the classifying locale of prime *ideals* of A. Its associated topological space of points is the set of prime ideals of A equipped with the *constructible topology*.

In an intuitionistic context, any of the (generalized) spaces of items 1–4 can be adopted as sensible definitions of the spectrum of A. Item 5 is then a tautology. The classical definition of the spectrum as a topological space doesn't work very well, because verifying the universal property one expects of it requires the Boolean Prime Ideal Theorem. Most dramatically, there are rings which are not trivial yet have neither prime ideals nor prime filters. The classical definition yields in this case the empty space.

First steps in the little Zariski topos

Let A be a ring. Let $\mathfrak{f}_0$ be the **generic prime filter** of A; it is a subobject of the constant sheaf $\underline{A}$ of the little Zariski topos.

- The ring $A^\sim := \underline{A}[\mathfrak{f}_0^{-1}]$ is the generic local localization of A.
- Given an A-module M, we have the $A^\sim$-module $M^\sim := \underline{M}[\mathfrak{f}_0^{-1}]$.

The Kripke–Joyal semantics for the little Zariski topos amounts to the following: $\mathrm{Spec}(A) \models \varphi$ iff $D(1) \models \varphi$, and the clauses for $D(f) \models \varphi$, where f ranges over the elements of A, are given by the following table.

$D(f) \models \top$	iff	true
$D(f) \models \bot$	iff	f is nilpotent
$D(f) \models x = y$	iff	$x = y \in M[f^{-1}]$
$D(f) \models \varphi \wedge \psi$	iff	$D(f) \models \varphi$ and $D(f) \models \psi$
$D(f) \models \varphi \vee \psi$	iff	there exists a partition $f^n = fg_1 + \cdots + fg_m$ with, for each i, $D(fg_i) \models \varphi$ or $D(fg_i) \models \psi$
$D(f) \models \varphi \Rightarrow \psi$	iff	for all $g \in A$, $D(fg) \models \varphi$ implies $D(fg) \models \psi$
$D(f) \models \forall x : M^\sim .\, \varphi(x)$	iff	for all $g \in A$ and $x_0 \in M[(fg)^{-1}]$, $D(fg) \models \varphi(x_0)$
$D(f) \models \exists x : M^\sim .\, \varphi(x)$	iff	there exists a partition $f^n = fg_1 + \cdots + fg_m$ with, for each i, $D(fg_i) \models \varphi(x_0)$ for some $x_0 \in M[(fg_i)^{-1}]$

The generic prime filter $\mathfrak{f}_0$ can also be described in explicit terms. For ring elements f and s, $D(f) \models (s \in \mathfrak{f}_0)$ iff $f \in \sqrt{(s)}$.

First steps in the little Zariski topos

Let A be a ring. Let $\mathfrak{f}_0$ be the **generic prime filter** of A; it is a subobject of the constant sheaf $\underline{A}$ of the little Zariski topos.

- The ring $A^\sim := \underline{A}[\mathfrak{f}_0^{-1}]$ is the generic local localization of A.
- Given an A-module M, we have the $A^\sim$-module $M^\sim := \underline{M}[\mathfrak{f}_0^{-1}]$.

> **Definition.** Let A be a ring and let M be an A-module. We define the *sheaf associated* to M on Spec A, denoted by $\tilde{M}$, as follows. For each prime ideal $\mathfrak{p} \subseteq A$, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} A$ we define the group $\tilde{M}(U)$ to be the set of functions $s\colon U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$, and such that s is locally a fraction m/f with $m \in M$ and $f \in A$. To be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and there are elements $m \in M$ and $f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. We make $\tilde{M}$ into a sheaf by using the obvious restriction maps.

Robin Hartshorne. Algebraic Geometry. 1977.

Our description of $M^\sim$ reveals a precise sense in which $M^\sim$ and M are related: $M^\sim$ is simply a localization of M (first lifted to another universe by the constant sheaf construction). The classical descriptions don't make the relation evident.

As a first approximation, the module $M^\sim$ can be thought of as a reification of all the stalks of M as a single object. The metatheorem displayed at the top left on the next slides makes this precise and also exposes the limits of this view: It is only correct for geometric sequents. When considering nongeometric sequents, phenomena appear which are unique to $M^\sim$ in the sense that they are in general not shared by M, its stalks or its quotients.

First steps in the little Zariski topos

Let A be a ring. Let $\mathfrak{f}_0$ be the **generic prime filter** of A; it is a subobject of the constant sheaf $\underline{A}$ of the little Zariski topos.

- The ring $A^\sim := \underline{A}[\mathfrak{f}_0^{-1}]$ is the generic local localization of A.
- Given an A-module M, we have the $A^\sim$-module $M^\sim := \underline{M}[\mathfrak{f}_0^{-1}]$.

Assuming the Boolean prime ideal theorem, a geometric sequent "$\forall \ldots \forall. (\cdots \Longrightarrow \cdots)$", where the two subformulas may not contain "$\Rightarrow$" and "$\forall$", holds for $M^\sim$ iff it holds for all stalks $M_\mathfrak{p}$.

$M^\sim$ inherits any property of M which is **localization-stable**.

If A is reduced ($x^n = 0 \Rightarrow x = 0$):

$A^\sim$ is a **field** (nonunits are zero).
$A^\sim$ has **¬¬-stable equality**.
$A^\sim$ is **anonymously Noetherian**.

One can show, assuming that the little Zariski topos is *overt*, that the module M in Set and the module $\underline{M}$ of the little Zariski topos share all first-order properties. This observation explains the metatheorem displayed at the bottom left. The assumption is satisfied if any element of A is nilpotent or not nilpotent, so it's always satisfied if the law of excluded middle is available in the metatheory. In an intuitionistic context, it's still "morally satisfied". Details are in Section 12.9 of these notes.

The metatheorem soups up a number of lemmas of algebraic geometry, there stated in geometric language. For instance, if M is finitely generated, then $M^\sim$ is of finite type. If M is finitely presented, then $M^\sim$ is of finite presentation. If M is coherent, then $M^\sim$ is coherent.

As an aside, the little Zariski topos is rarely Boolean (validates the law of excluded middle). A necessary condition is that A is of dimension ≤ 0.

First steps in the little Zariski topos

Let A be a ring. Let $\mathfrak{f}_0$ be the generic prime filter of A; it is a subobject of the constant sheaf $\underline{A}$ of the little Zariski topos.

ON THE SPECTRUM OF A RINGED TOPOS **209**

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec(**E**, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

Assuming that A is reduced, the following nongeometric sequents hold in the little Zariski topos (among others):

$A^\sim$ is a field in the sense that zero is the only noninvertible element. This field property was already observed in the 1970s by Mulvey, who didn't know a deeper reason for this property. We now know that it's a shadow of an internal property whose external translation expresses that $A^\sim$ is quasicoherent.

$A^\sim$ has $\neg\neg$-stable equality in the sense that

$$\mathrm{Spec}(A) \models \forall s : A^\sim.\ \neg\neg(s = 0) \Rightarrow s = 0.$$

Classically, every set has $\neg\neg$-stable equality; intuitionistically, this is a special property of some sets. It's quite useful, as some theorems of classical commutative algebra can only be proven intuitionistically when weakened by double negation. The stability then allows, in some cases, to obtain the original conclusion.

$A^\sim$ is anonymously Noetherian in the sense that any of its ideals is *not not* finitely generated. A philosophically-motivated constructivist might be offended by this notion, since it runs counter to the maxim that constructive mathematics should be informative. However, in the internal context it is a useful notion: Hilbert's basis theorem holds for it, and we'll put it to good use in our proof of Grothendieck's generic freeness lemma.

First steps in the little Zariski topos

Let A be a ring. Let f_0 be the generic prime filter of A; it is a

Complexity reduction

The external meaning of

$$\mathrm{Spec}(A) \models \ulcorner A^{\sim}[X_1,\ldots,X_n] \text{ is anonymously Noetherian} \urcorner$$

is:

For any element $f \in A$ and any (not necessarily quasicoherent) sheaf of ideals $\mathcal{J} \hookrightarrow A^{\sim}[X_1,\ldots,X_n]|_{D(f)}$: If
  for any element $g \in A$ the condition that
    the sheaf $\mathcal{J}$ is of finite type on $D(g)$
  implies that $g = 0$,
then $f = 0$.

Are there theorems which can only be proven using the internal language and not be proven without?

No. Just as the translation from internal statements to external statements is entirely mechanical, so is the translation from internal proofs to external proofs. Any proof employing the internal language can be unwound to yield an external proof not referencing the internal language.

However, depending on the logical complexity of the statements occurring in a given proof, the resulting external proof might be (much) more complex than the internal proof. This is particularly the case if the proof involves double negation, for much the same reason as that in computer science, continuations can twist the control flow in nontrivial ways which are sometimes hard to understand. It is in these cases where we can extract the most value of the internal language, unlocking notions and proofs which might otherwise be hard to obtain.

The slide shows a specific example. The internal statement that $A^{\sim}[X_1,\ldots,X_n]$ is anonymously Noetherian is quite simple; its external translation is quite convoluted.

Revisiting the test cases

Let A be a reduced ring.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

A baby application

Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

Proof. The matrix is surjective over the field $A^\sim$. This is a contradiction to basic linear algebra. Hence $\mathrm{Spec}(A) \models \bot$, thus $1 = 0$ in A.

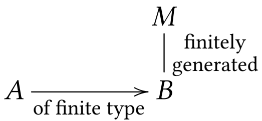
A child application

Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

Proof. The matrix is injective over the field $A^\sim$. This is a contradiction to basic linear algebra. Hence $\mathrm{Spec}(A) \models \bot$, thus $1 = 0$ in A.

This slide delivers on the promise made earlier: Using the internal language of the little Zariski topos, we can reduce to the case of fields without having to employ maximal ideals, prime ideals or minimal prime ideals. Since the internal language machinery is itself constructive, the displayed proofs can be unwound to yield external constructive proofs which don't reference topos theory. Details on these external proofs can soon be found in a forthcoming paper titled Without loss of generality, any reduced ring is a field.

Revisiting the test cases



Let A be a reduced ring.

Generic freeness

Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
Let M be a finitely generated B-module ($\cong B^m/U$).
If $f = 0$ is the only element of A such that

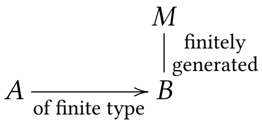
1. $B[f^{-1}]$ and $M[f^{-1}]$ are free modules over $A[f^{-1}]$,
2. $A[f^{-1}] \to B[f^{-1}]$ is of finite presentation and
3. $M[f^{-1}]$ is finitely presented as a module over $B[f^{-1}]$,

then $1 = 0$ in A.

The test case of Grothendieck's generic freeness lemma illustrates that the internal language of toposes can help commutative algebra even if one is not interested in constructivity issues. The previously known proofs are somewhat long and somewhat convoluted; the new proof is arguably short and simple.

Details on the internal proof are in Section 11.5 of these notes. The external proof obtained by unwinding the internal one is still quite direct, compared to the previously published proofs, and interestingly follows a curious course: It starts with verifying, in an inductive manner, that B and M are free; that $A \to B$ is of finite presentation; and that M is finitely presented as a B-module. Then the assumption for $f = 1$ is used, rendering the prior steps moot, since over the zero ring any module is free and any algebra is of finite presentation. External translations of internal proofs which use double negation will always take such a course.

Revisiting the test cases



Let A be a reduced ring.

Generic freeness

Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
Let M be a finitely generated B-module ($\cong B^m/U$).
If $f = 0$ is the only element of A such that

1. $B[f^{-1}]$ and $M[f^{-1}]$ are free modules over $A[f^{-1}]$,
2. $A[f^{-1}] \to B[f^{-1}]$ is of finite presentation and
3. $M[f^{-1}]$ is finitely presented as a module over $B[f^{-1}]$,

then $1 = 0$ in A.

Proof. In the little Zariski topos it's **not not** the case that

1. $B^\sim$ and $M^\sim$ are free modules over $A^\sim$,
2. $A^\sim \to B^\sim$ is of finite presentation and
3. $M^\sim$ is finitely presented as a module over $B^\sim$,

by basic linear algebra over the field $A^\sim$. The claim is precisely the external translation of this fact.

Here's a rough sketch why the double negations occur. In undergraduate linear algebra, we learn that any finitely generated vector space is free (has a basis), by the following argument: Let $(v_1, \ldots, v_n)$ be a generating family. Either one of the generators can be expressed as a linear combination of the others or not. In the latter case, the family is linearly independent and we're done. In the former, we remove the redundant generator and continue by induction.

This argument relies on the law of excluded middle and can therefore not put to work in the internal universe as is. However, intuitionistically we do have the weaker statement $\neg\neg(\varphi \vee \neg\varphi)$, and we can use this law in place of the law of excluded middle to intuitionistically deduce that any finitely generated vector space does *not not* have a basis.

Understanding algebraic geometry

Understand **notions of algebraic geometry** over a scheme X as **notions of algebra** internal to $\mathrm{Sh}(X)$.

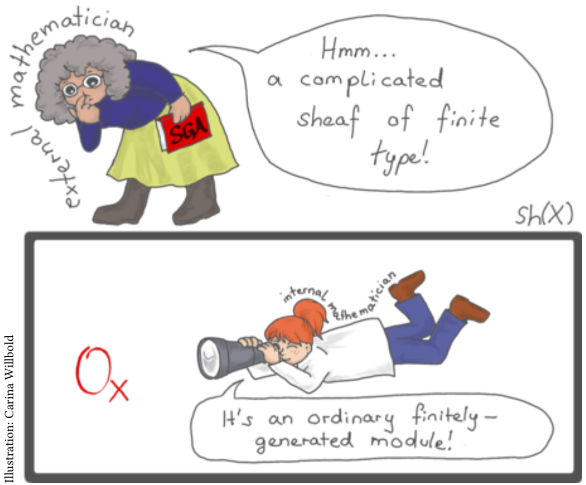


Illustration: Carina Willbold

One doesn't need to be an expert in topos theory in order to know that many notions in algebraic geometry are inspired by notions in algebra and that proofs in algebraic geometry often proceed by reducing to algebra. The internal language is a way of making this connection precise: In many cases, the former are simply interpretations of the latter internal to $\mathrm{Sh}(X)$. Because this connection is precise instead of informal, additional value is gained: We can skip many basic proofs in algebraic geometry because they're just externalizations of proofs in algebra carried out internally to $\mathrm{Sh}(X)$.

Understanding algebraic geometry

Understand **notions of algebraic geometry** over a scheme X as **notions of algebra** internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set
sheaf of modules	module
sheaf of finite type	finitely generated module
tensor product of sheaves	tensor product of modules
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$
dimension of X	Krull dimension of $\mathcal{O}_X$
spectrum of a sheaf of $\mathcal{O}_X$-algebras	ordinary spectrum [with a twist]
big Zariski topos of X	big Zariski topos of the ring $\mathcal{O}_X$ [with a twist]
higher direct image	sheaf cohomology

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

$\Longleftarrow$

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

A basic example is as follows. A short exact sequence of sheaves of modules looks like a short exact sequence of plain modules from the internal point of view of $\mathrm{Sh}(X)$. If the two outer sheaves are of finite type, then from the internal point of view, the two outer modules will look like finitely generated modules. Because the standard proof of the proposition quoted on the lower right is intuitionistically valid, it follows that, from the internal point of view, the middle module is too finitely generated. Consulting the dictionary a second time, this amounts to saying that the middle sheaf is of finite type.

More details on this research program can be found in these notes, partly reported on at the 2015 IHÉS conference. Even though many important dictionary entries are still missing (for instance pertaining to derived categories and intersection theory), I believe that it is already at its current stage useful to working algebraic geometers.

The next few slides illustrate an entirely different way of approaching algebraic geometry using topos theory.

Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  set of prime ideals of $\mathbb{Z}[X,Y,Z]/(X^n+Y^n-Z^n)$ + Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring → Set.

$$A \longmapsto \{(x,y,z)\in A^3 \mid x^n+y^n-z^n=0\}$$

At the Secret Blogging Seminar, there was an insightful long-running discussion on the merits of the two approaches. Two disadvantages of the approach using locally ringed spaces is that the underlying topological spaces don't actually parametrize "honest", "geometric" points, but the more complex notion of irreducible closed subsets; and that they don't work well in a constructive setting. (For this, they would have to be replaced by locally ringed locales.)

The functorial approach is more economical, philosophically rewarding, and works constructively. Given a functor $F : \mathrm{Ring} \to \mathrm{Set}$, we imagine $F(A)$ to be the set of "A-valued points" of the hypothetical scheme described by F, the set of "points with coordinates in A". These sets have direct geometric meaning. However, typically only field-valued points are easy to describe. For instance, the functor representing projective n-space is given on fields by

$$K \longmapsto \text{the set of lines through the origin in } K^{n+1} \cong \{[x_0 : \cdots : x_n] \mid x_i \neq 0 \text{ for some } i\},$$

whereas on general rings it is given by

$$A \longmapsto \text{the set of quotients } A^{n+1} \twoheadrightarrow P, \text{ where } P \text{ is projective, modulo isomorphism.}$$

It is these more general kinds of points which impart a meaningful sense of cohesion on the field-valued points, so they can't simply be dropped from consideration.

Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  set of prime ideals of $\mathbb{Z}[X,Y,Z]/(X^n+Y^n-Z^n)$ + Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

$$A \longmapsto \{(x,y,z) \in A^3 \mid x^n + y^n - z^n = 0\}$$

Synthetic approach: model schemes **directly as sets** in the internal universe of the **big Zariski topos** of a base scheme.

$$\{(x,y,z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

This tension is resolved by observing that the category of functors Ring $\to$ Set is a topos (the *big Zariski topos* of Spec($\mathbb{Z}$)) and that we can therefore employ its internal language. This language takes care of juggling stages behind the scenes. For instance, projective n-space can be described by the naive expression

$$\{(x_0,\ldots,x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\underline{\mathbb{A}}^1)^\times.$$

This example illustrates our goal: to develop a synthetic account of algebraic geometry, in which schemes are plain sets and morphisms between schemes are maps between those sets. It turns out that there are many similarities with the well-developed synthetic account of differential geometry, but also important differences.

The big Zariski topos

Let S be a fixed base scheme.

Definition

The **big Zariski topos** $\mathrm{Zar}(S)$ of a scheme S is equivalently

1. the topos of sheaves over $(\mathrm{Aff}/S)_{\mathrm{lofp}}$,
2. the classifying topos of local rings over S or
3. the classifying $\mathrm{Sh}(S)$-topos of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.

- For an S-scheme X, its functor of points $\underline{X} = \mathrm{Hom}_S(\cdot, X)$ is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- In particular, there is the ring object $\underline{\mathbb{A}}^1$ with $\underline{\mathbb{A}}^1(T) = \mathcal{O}_T(T)$.
- This ring object is a **field**: nonzero implies invertible. [Kock 1976]

The objects of the category $(\mathrm{Aff}/S)_{\mathrm{lofp}}$ are morphisms of the form $\mathrm{Spec}(R) \to S$ which are locally of finite presentation. (Other choices of resolving set-theoretical issues of size are also possible.)

A functor $F : (\mathrm{Aff}/S)^{\mathrm{op}}_{\mathrm{lofp}} \to \mathrm{Set}$ is a sheaf for the Zariski topology if and only if the diagram

$$F(T) \to \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

is a limit diagram for any open covering $T = \bigcup_i U_i$ of any scheme $T \in (\mathrm{Aff}/S)_{\mathrm{lofp}}$.

In the case that $S = \mathrm{Spec}(A)$ is affine, the big Zariski topos of S is also simply called "big Zariski topos of A". It is a subtopos of the topos of functors $\mathrm{Alg}(A) \to \mathrm{Set}$ and classifies local A-algebras.

From the internal point of view of $\mathrm{Sh}(S)$, the sheaf $\mathcal{O}_S$ of rings is just an ordinary ring, and we can construct internally to $\mathrm{Sh}(S)$ the big Zariski topos of $\mathcal{O}_S$. Externally, this construction will yield a certain bounded topos over $\mathrm{Sh}(S)$. However, as indicated on the slide, this topos will *not* coincide with the true big Zariski topos of S. To construct the true big Zariski topos, we have to build, internally to $\mathrm{Sh}(S)$, the classifying topos of local *and local-over-$\mathcal{O}_S$* $\mathcal{O}_S$-algebras.

Synthetic constructions

$\mathbb{A}^n = (\underline{\mathbb{A}}^1)^n = \underline{\mathbb{A}}^1 \times \cdots \times \underline{\mathbb{A}}^1$

$\mathbb{P}^n = \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\underline{\mathbb{A}}^1)^\times$
$\cong$ set of one-dimensional subspaces of $(\underline{\mathbb{A}}^1)^{n+1}$
(with $\mathcal{O}(-1) = (\ell)_{\ell : \mathbb{P}^n}$, $\mathcal{O}(1) = (\ell^\vee)_{\ell : \mathbb{P}^n}$)

Spec(R) $= \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1) =$ set of $\underline{\mathbb{A}}^1$-valued points of R

$TX = X^\Delta$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$

A subset $U \subseteq X$ is **qc-open** if and only if for any $x : X$ there exist $f_1, \ldots, f_n : \underline{\mathbb{A}}^1$ such that $x \in U \iff \exists i.\, f_i \neq 0$.

A **synthetic affine scheme** is a set which is in bijection with $\mathrm{Spec}(R)$ for some synthetically quasicoherent $\underline{\mathbb{A}}^1$-algebra R.

A **finitely presented synthetic scheme** is a set which can be covered by finitely many qc-open f.p. synthetic affine schemes U_i such that the intersections $U_i \cap U_j$ can be covered by finitely many qc-open f.p. synthetic affine schemes.

In the internal universe of the big Zariski topos of a base scheme S, S-schemes can simply be modeled by sets (enjoying the special property that, in a certain precise sense, they are locally affine). This slide expresses some of the basic constructions of S-schemes in that language.

Particularly nice are the following items.

- Projective n-space can be given by the any of the two quite naive expressions displayed on the slide.
- Let X be an S-scheme. We often think about a sheaves of $\mathcal{O}_X$-modules over X by their fibers; but for a rigorous treatment in the standard foundations, we have to take the full sheaf structure into account; the fibers do not determine a sheaf uniquely.

  From the internal point of view of $\mathrm{Zar}(S)$, a sheaf of $\mathcal{O}_X$-modules is indeed simply a family of $\underline{\mathbb{A}}^1$-modules, one $\underline{\mathbb{A}}^1$-module for each element of $\underline{X}$. The slide illustrates how we can define the Serre twisting sheaves in this language.

Synthetic constructions

$\mathbb{A}^n = (\underline{\mathbb{A}}^1)^n = \underline{\mathbb{A}}^1 \times \cdots \times \underline{\mathbb{A}}^1$

$\mathbb{P}^n = \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\underline{\mathbb{A}}^1)^\times$
$\cong$ set of one-dimensional subspaces of $(\underline{\mathbb{A}}^1)^{n+1}$
(with $\mathcal{O}(-1) = (\ell)_{\ell : \mathbb{P}^n}$, $\mathcal{O}(1) = (\ell^\vee)_{\ell : \mathbb{P}^n}$)

$\mathrm{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ = set of $\underline{\mathbb{A}}^1$-valued points of R

$TX = X^\Delta$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$

A subset $U \subseteq X$ is **qc-open** if and only if for any $x : X$ there exist $f_1, \ldots, f_n : \underline{\mathbb{A}}^1$ such that $x \in U \iff \exists i.\, f_i \neq 0$.

A **synthetic affine scheme** is a set which is in bijection with $\mathrm{Spec}(R)$ for some synthetically quasicoherent $\underline{\mathbb{A}}^1$-algebra R.

A **finitely presented synthetic scheme** is a set which can be covered by finitely many qc-open f.p. synthetic affine schemes U_i such that the intersections $U_i \cap U_j$ can be covered by finitely many qc-open f.p. synthetic affine schemes.

- The spectrum of an $\underline{\mathbb{A}}^1$-algebra can be given by the naive expression displayed on the slide. It looks like this expression can't be right, ignoring any non-maximal ideals; however, it is.
- The big Zariski topos of an S-scheme X is, from the internal point of view of $\mathrm{Zar}(S)$, simply the slice topos $\mathrm{Set}/\underline{X}$. Hence to give an X-scheme simply amounts to giving an $\underline{X}$-indexed family of sets.

Synthetic algebraic geometry has been developed up to the point of étale geometric morphisms. Much remains to be done: For instance, as of yet there is only an account of Čech methods for computing cohomology, there is not yet a synthetic treatment of true cohomology. Derived categories and intersection theory are also missing.

Relations between the Zariski toposes

The big Zariski topos is a topos over the small Zariski topos:

$$\begin{array}{rrcl} \pi: & \mathrm{Zar}(A) & \longrightarrow & \mathrm{Spec}(A) \\ & \text{local } A\text{-algebra } (A \xrightarrow{\alpha} B) & \longmapsto & (A \to A[(\alpha^{-1}[B^\times])^{-1}]) \end{array}$$

This morphism is **connected** (π^{-1} is fully faithful) and **local**, so there is a preinverse

$$\begin{array}{rcl} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Zar}(A) \\ \text{local localization } (A \to B) & \longmapsto & (A \to B) \end{array}$$

which is a subtopos inclusion inducing an idempotent monad $\sharp$ and an idempotent comonad $\flat$ on $\mathrm{Zar}(S)$.

- Internally to $\mathrm{Zar}(S)$, $\mathrm{Spec}(S)$ can be constructed as the **largest subtopos** where $\flat\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is bijective.
- Internally to $\mathrm{Spec}(S)$, $\mathrm{Zar}(S)$ can be constructed as the **classifying topos** of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.
- $\mathrm{Zar}(A)$ is the **lax pullback** $(\mathrm{Set} \Rightarrow_{\mathrm{Set}[\mathrm{Ring}]} \mathrm{Set}[\mathrm{LocRing}])$.

Let A be a ring. By definition, we obtain a geometric morphism $\mathrm{Set} \to \mathrm{Set}[\mathrm{Ring}]$ into the classifying topos of rings. There is also a geometric morphism $\mathrm{Set}[\mathrm{LocRing}] \to \mathrm{Set}[\mathrm{Ring}]$, obtained by realizing that any local ring is in particular a ring. These morphisms fit together in a lax pullback square as follows:

$$\begin{array}{ccc} \mathrm{Zar}(A) & \longrightarrow & \mathrm{Set}[\mathrm{LocRing}] \\ \downarrow & \nearrow\!\!\!\nearrow & \downarrow \\ \mathrm{Set} & \longrightarrow & \mathrm{Set}[\mathrm{Ring}] \end{array}$$

This observation is joint with Peter Arndt and Matthias Hutzler.

Incidentally, the pseudo pullback of the morphism $\mathrm{Set}[\mathrm{LocRing}] \to \mathrm{Set}[\mathrm{Ring}]$ along $\mathrm{Set} \to \mathrm{Set}[\mathrm{Ring}]$ is not very interesting: It's the largest subtopos of Set where A is a local ring. Assuming the law of excluded middle, this subtopos is either the trivial topos (if A is not local) or Set (if A is local).

There is also a way of realizing the little Zariski topos of A as a pseudo pullback, exploiting that the (localic) spectrum construction is geometric. See Section 12.6 of these notes for details.

Properties of the affine line

- $\underline{\mathbb{A}}^1$ is a field:

$$\neg(x = 0) \iff x \text{ invertible} \qquad \text{[Kock 1976]}$$
$$\neg(x \text{ invertible}) \iff x \text{ nilpotent}$$

- $\underline{\mathbb{A}}^1$ satisfies the axiom of microaffinity: Any map $f : \Delta \to \underline{\mathbb{A}}^1$ is of the form $f(\varepsilon) = a + b\varepsilon$ for unique values $a, b : \underline{\mathbb{A}}^1$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$.

- Any map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial function.

- $\underline{\mathbb{A}}^1$ is anonymously algebraically closed: Any monic polynomial does *not not* have a zero.

The axiom of microaffinity is a special instance of the *Kock–Lawvere axiom* known from synthetic differential geometry. We'll see on the next slide that $\underline{\mathbb{A}}^1$ validates an unusually strong form of the Kock–Lawvere axiom, not at all satisfied in the usual well-adapted models of synthetic differential geometry.

The fact that, internally to $\mathrm{Zar}(S)$, any map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial can be seen as a formal version of the general motto that in algebraic geometry, "morphisms are polynomials".

Synthetic quasicoherence

Recall $\mathrm{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ and consider the statement

$$\text{“the canonical map}\quad \begin{array}{rcl} R & \longrightarrow & (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(R)} \\ f & \longmapsto & (\alpha \mapsto \alpha(f)) \end{array} \quad \text{is bijective”.}$$

- True for $R = \underline{\mathbb{A}}^1[X]/(X^2)$ (microaffinity).
- True for $R = \underline{\mathbb{A}}^1[X]$ (every function is a polynomial).
- True for **any** finitely presented $\underline{\mathbb{A}}^1$-algebra R.

Any known property of $\underline{\mathbb{A}}^1$ follows from this **synthetic quasicoherence**.

the mystery of nongeometric sequents

Let R be an $\underline{\mathbb{A}}^1$-algebra. An element $f \in R$ induces an $\underline{\mathbb{A}}^1$-valued function on $\mathrm{Spec}(R)$; functions of this form can reasonably be called “algebraic”. In a synthetic context, there should be no other $\underline{\mathbb{A}}^1$-valued functions on $\mathrm{Spec}(R)$ as these algebraic ones, and different algebraic expressions should yield different functions. This is precisely what the bijectivity of the displayed map expresses (in a positive way).

In synthetic differential geometry, the closest cousin of synthetic algebraic geometry, the analogue of the displayed map is only bijective for Weil algebras such as $\underline{\mathbb{A}}^1[X]/(X^2)$ or $\underline{\mathbb{A}}^1[X, Y]/(X^2, XY)$, not for arbitrary finitely presented $\underline{\mathbb{A}}^1$-algebras. This is a major difference to synthetic differential geometry.

Synthetic quasicoherence

Recall $\mathrm{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ and consider the statement

$$\text{“the canonical map}\quad \begin{array}{rcl} R & \longrightarrow & (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(R)} \\ f & \longmapsto & (\alpha \mapsto \alpha(f)) \end{array} \quad \text{is bijective”.}$$

- True for $R = \underline{\mathbb{A}}^1[X]/(X^2)$ (microaffinity).
- True for $R = \underline{\mathbb{A}}^1[X]$ (every function is a polynomial).
- True for **any** finitely presented $\underline{\mathbb{A}}^1$-algebra R.

Any known property of $\underline{\mathbb{A}}^1$ follows from this **synthetic quasicoherence**.

the mystery of nongeometric sequents

The notion of synthetic quasicoherence is interesting for a number of reasons:

- All currently known properties of $\underline{\mathbb{A}}^1$, such as all the properties listed on the previous slide, follow from the statement that $\underline{\mathbb{A}}^1$ is synthetically quasicoherent.

  For instance, here is how we can verify the field property. Let $x : \underline{\mathbb{A}}^1$ such that $x \neq 0$. Set $R = \underline{\mathbb{A}}^1/(x)$. Then $\mathrm{Spec}(R) = \emptyset$. Thus $(\underline{\mathbb{A}}^1)^{\mathrm{Spec}(R)}$ is a singleton. Hence $R = 0$. Therefore x is invertible.
- Given an $\underline{\mathbb{A}}^1$-module E, we can formulate the following variant of the axiom of synthetic quasicoherence: “For any finitely presented $\underline{\mathbb{A}}^1$-algebra R, the canonical map $R \otimes_{\underline{\mathbb{A}}^1} E \to E^{\mathrm{Spec}(R)}$ is bijective.” This axiom is satisfied if and only if E is induced by a quasicoherent sheaf of $\mathcal{O}_S$-modules.
- The notion of synthetic quasicoherence is central to synthetic algebraic geometry. The notions of synthetic open immersions, closed immersion, schemes and several others all refer to synthetic quasicoherence.

Synthetic quasicoherence

Recall $\mathrm{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ and consider the statement

$$\text{“the canonical map} \quad \begin{array}{rcl} R & \longrightarrow & (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(R)} \\ f & \longmapsto & (\alpha \mapsto \alpha(f)) \end{array} \quad \text{is bijective”.}$$

- True for $R = \underline{\mathbb{A}}^1[X]/(X^2)$ (microaffinity).
- True for $R = \underline{\mathbb{A}}^1[X]$ (every function is a polynomial).
- True for **any** finitely presented $\underline{\mathbb{A}}^1$-algebra R.

Any known property of $\underline{\mathbb{A}}^1$ follows from this **synthetic quasicoherence**.

the mystery of nongeometric sequents

An analogue of synthetic quasicoherence holds in the classifying topos of rings, demonstrating that even presheaf toposes can validate interesting nontrivial nongeometric sequents.

We believe that an analogue of synthetic quasicoherence holds for the generic model of any geometric theory. This is work in progress. If true, this would yield a major source of nongeometric sequents in classifying toposes. Because of the many applications on nongeometric sequents, it's very desirable to possess such a source.

The mystery of nongeometric sequents is this: On the one hand, they are very useful to have because of surprising applications; on the other hand, they are as of yet quite elusive.

Classifying toposes in algebraic geometry

(big) topos	classified theory
Zariski	local rings [Hakim 1972]
étale	separably closed local rings [Hakim 1972, Wraith 1979]
fppf	fppf-local rings (conjecturally: algebraically closed local rings)
ph	?? (conjecturally: algebraically closed valuation rings validating the projective Nullstellensatz)
surjective	algebraically closed geometric fields
$\neg\neg$	?? (conjecturally: algebraically closed geometric fields which are integral over the base)
infinitesimal	local algebras together with a nilpotent ideal [Hutzler 2018]
crystalline	??

Toposes, and also more specifically classifying toposes, originated in algebraic geometry. It is therefore deeply embarrassing that as of now, still very little is known about the theories classified by the major toposes in active use by algebraic geometers.

> It is my belief that a great many of the toposes occuring in algebraic geometry can be conveniently described in terms of the theories they classify. This is certainly so in the case of étale and crystalline toposes, for example.
>
> Since a ring of polynomials in many variables is finitely presented and faithfully flat over its subring of symmetric polynomials, one may deduce that the inverse image of the generic commutative ring in the f.p.p.f. topos has the property that monic polynomials over it split into linear factors. This leads me to conjecture that the f.p.p.f. topos classifies algebraically closed local rings in an appropriate sense, but this remains a speculation for the present.

Gavin Wraith. Generic Galois theory of local rings. 1979.
(A cult classic and must-read for anyone interested in the intersection of topos theory and algebraic geometry.)

Classifying toposes in algebraic geometry

(big) topos	classified theory
Zariski	local rings [Hakim 1972]
étale	separably closed local rings [Hakim 1972, Wraith 1979]
fppf	fppf-local rings (conjecturally: algebraically closed local rings)
ph	?? (conjecturally: algebraically closed valuation rings validating the projective Nullstellensatz)
surjective	algebraically closed geometric fields
¬¬	?? (conjecturally: algebraically closed geometric fields which are integral over the base)
infinitesimal	local algebras together with a nilpotent ideal [Hutzler 2018]
crystalline	??

For almost forty years, only the big Zariski topos and its étale subtopos were understood in that way. These 2017 notes answer the question for the fppf topology and the surjective topology (in Section 21) and state conjectures for the ph topology and the double negation topology. However, while good to have, the answer for the fppf topology remains unsatisfactory, since Wraith's conjecture that the fppf topos classifies the simpler theory of algebraically closed local rings has neither been confirmed nor refuted.

A couple of weeks ago, Matthias Hutzler managed to determine the theory classified by the big infinitesimal topos of a ring A: It classifies pairs $(B, \mathfrak{a})$ consisting of a local A-algebra B and a nilpotent ideal $\mathfrak{a} \subseteq B$. Details will be in his forthcoming Master's thesis. He is currently working on answering the question for the closely related big crystalline topos.

It will be exciting to learn what the crystalline topos and the many other toposes in algebraic geometry classify; how algebraic geometry can profit from these discoveries; and which new flavors of synthetic algebraic geometry they unlock.