Regression analysis (in R)

Catalina Vallejos (The Alan Turing Institute and UCL)
Aaron Lun (Cancer Research UK - Cambridge Institute)

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Outline for the course

- **Introduction to statistics**
- **Hypothesis testing (in R)**
- **Regression analysis (in R)**

Regression analysis

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In regression, variables are clasified as

- **Response**: outcome variable of interest (dependent)
- **Explanatory**: covariate(s) to explain the response (independent)

Examples?

Regression analysis

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- For a continuous response (e.g. blood sugar level), a typical choice is **linear regression**

Today, we will focus on linear regression

Linear regression

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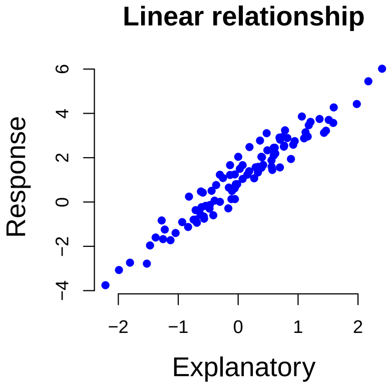
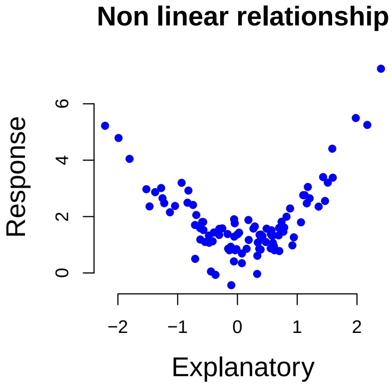

Linear relationship

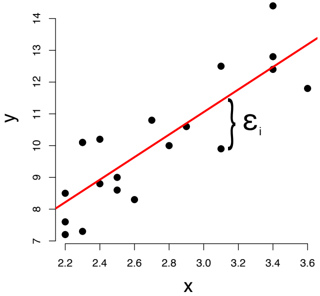

Non linear relationship

Simple linear regression

In **simple linear regression**, the aim is to capture a linear relationship between a response (y) and a **single** covariate (x)

The simple linear regression model is writen as

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad i = 1, \ldots, n$$

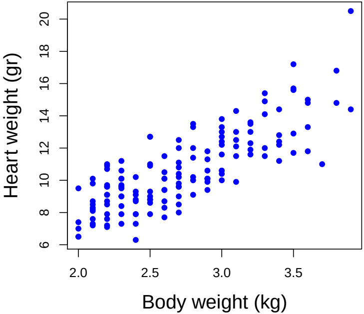


- β_0 is the intercept
- β_1 is the slope (gradient)
- ϵ_i is the error term

(*) Plot the response on the vertical axis and the covariate on the horizontal axis

Simple linear regression: an example

The data in Fisher (1947) shows heart and body weights for 144 cats



Is there a relationship between heart and body weight?

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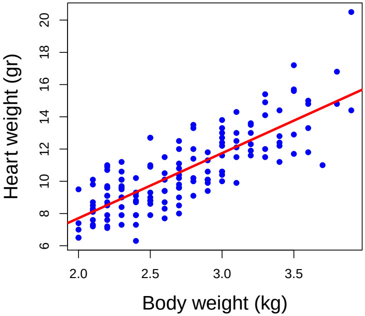

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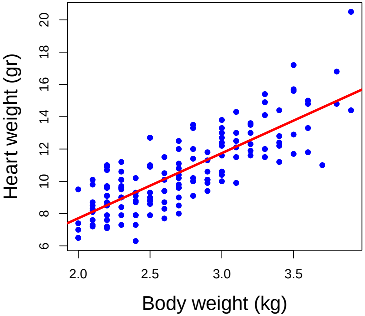

We can fit a simple linear regression

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where

- y_i = heart weight of cat i
- x_i = body weight of cat i

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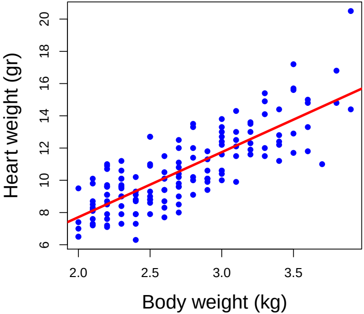

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How to estimate β_0 and β_1?

Simple linear regression

Idea: find β_0 and β_1 such that the line is a good fit for the data

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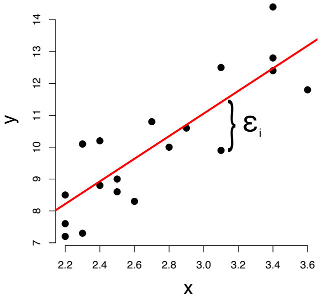

For example, by minimizing the **residuals** of the regression

$$\epsilon_i = Y_i - (\beta_0 + \beta_1 x_i) \equiv Y_i - \hat{Y}_i$$

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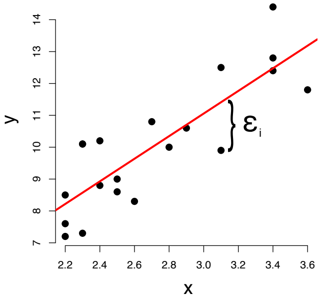

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Typically, this is done by minimizing the **Sum of Square Errors (SSE)**

$$\text{SSE} = \sum_{i=1}^{n} \epsilon_i^2$$

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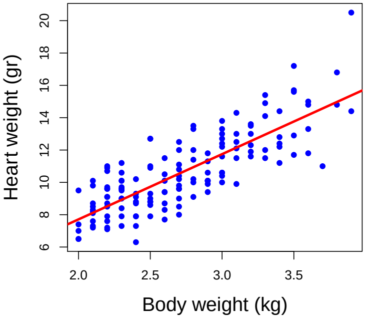

In this example we have

- β_0 = -0.36
- β_1 = 4.03

How to interpret this?

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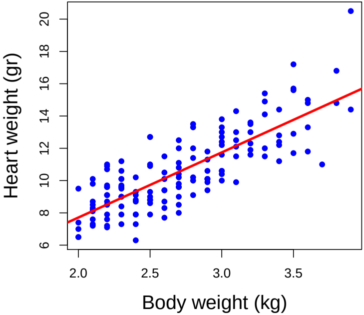

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For a 1kg increase in body weight, we expect heart weight to increase by 4.03gr

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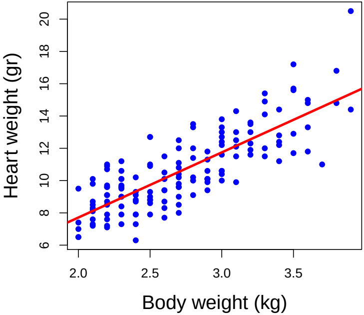

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For a 1kg increase in body weight, we expect heart weight to increase by 4.03gr

Is this increase statistically significant?

Simple linear regression

We can answer this using hypothesis testing:

$$H_0 : \beta_1 = 0 \quad \text{vs} \quad H_1 : \beta_1 \neq 0$$

Recall: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

→ if $\beta_1 = 0$, y_i and x_i are not (linearly) dependent

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→ if $\beta_1 = 0$, y_i and x_i are not (linearly) dependent

Assumming $\epsilon_i \sim \text{Normal}(0, \sigma^2)$, we can derive a *t*-test for β_1

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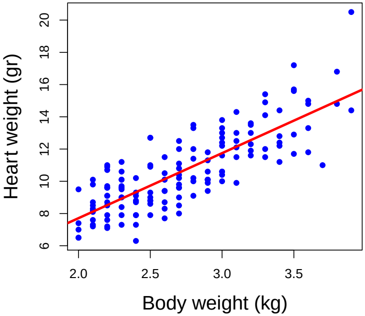

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The *p*-value is $< 2 \times 10^{-16}$

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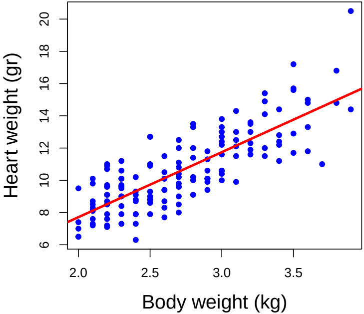

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We reject H_0, i.e. we conclude that the linear relationship between heart and body weight is statistically significant ($\alpha = 0.05$)

Simple linear regression: Analysis of Variance (ANOVA)

ANOVA aims to decompose the total variance of y

Recall: $\mathrm{Var}(y) = \sum_{i=1}^{n}(y_i - \bar{y})^2/(n-1)$, with $\bar{y} = (\sum_{i=1}^{n} y_i)/n$

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$$\underbrace{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}_{\text{SSR (Regression)}} = \underbrace{\sum_{i=1}^{n}(y_i - \bar{y})^2}_{\text{SST (Total)}} - \underbrace{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}_{\text{SSE (Error)}}$$

- SST quantifies the total variability of y
- SSE quantifies residual variability of y (unexplained by x)
- SSR quantifies how much of the variability of y is explained by x

Simple linear regression: Analysis of Variance (ANOVA)

The **coefficient of determination** R^2 is defined as the proportion of total variability of y that is explained by x

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}}$$

- $0 \leq R^2 \leq 1$
- If $R^2 = 0$, x explains none of the variability of y
- If $R^2 = 1$, x explains all of the variability of y (perfect fit!)

Warning: this assumes the relationship between x and y is linear

Simple linear regression: an example

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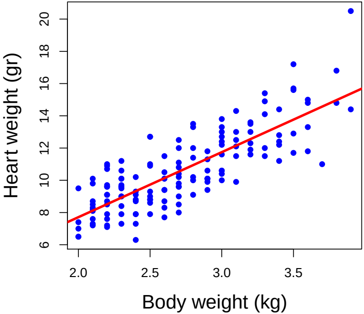

In this example we have

- $R^2 = 0.65$

Body weight explains 65% of the variability of a cat's heart weight

Simple linear regression: Analysis of Variance (ANOVA)

This variance decomposition is summarized by the **ANOVA table**

Source	**Degrees of freedom**	**Sum of Squares**	**Mean Square**	**F ratio**
Regression	1	SSR	SSR/1	$\frac{\text{SSR}}{1} \ / \ \frac{\text{SSE}}{n-2}$
Error	$n-2$	SSE	SSE/$(n-2)$	
Total	$n-1$	SST	SST/$(n-1)$	

- Assuming $\epsilon_i \sim \text{Normal}(0, \sigma^2)$, the F ratio can be used to test

$$H_0 : \beta_1 = 0 \text{ versus } H_1 : \beta_1 \neq 0$$

How does this compare to the t-test introduced earlier?

Simple linear regression versus correlation

Pearson's correlation ($\rho_{x,y}$) quantifies linear dependency between variables x and y

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Examples?

Let's play: `http://guessthecorrelation.com`

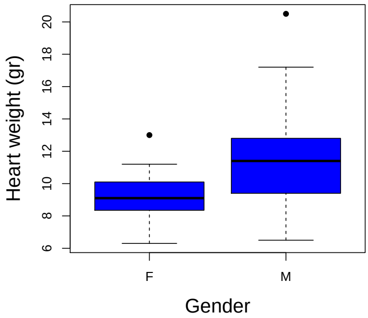
Questions + practical

http://bioinformatics.cruk.cam.ac.uk/apps/winter-school/linear/

Simple linear regression: categorical covariates

Simple linear regression: an example

The data in Fisher (1947) also contains gender information



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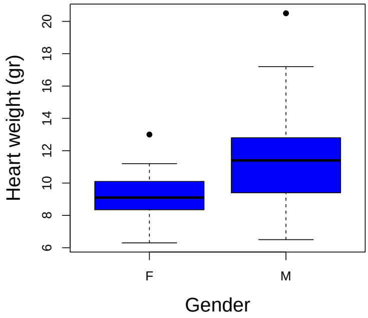

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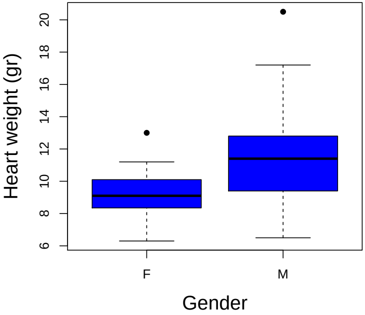

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In this example we have:

$$t = \text{-}5.35$$
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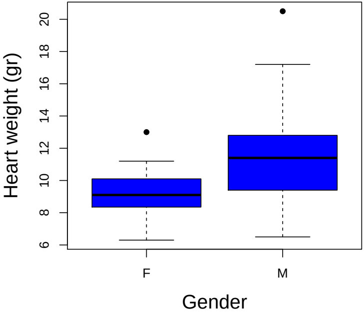

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Can we answer the same question using linear regression?

Simple linear regression: categorical covariates

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Recall: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

Instead of defining a single regression effect β_1, we need to estimate an effect for **each level** of the categorical covariate

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Instead of defining a single regression effect β_1, we need to estimate an effect for **each level** of the categorical covariate

This can be done using **dummy variables**

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For example,

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Note: gender has **two** levels, but we only define **one** dummy variable (here we left *female* as a **reference category**)

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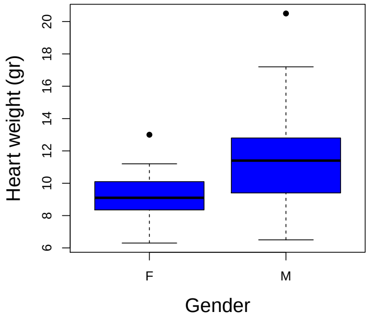

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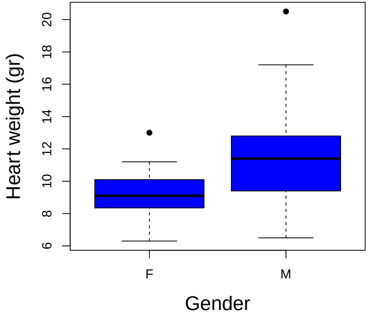

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- For a male cat: $\hat{y}_i = \beta_0 + \beta_1$
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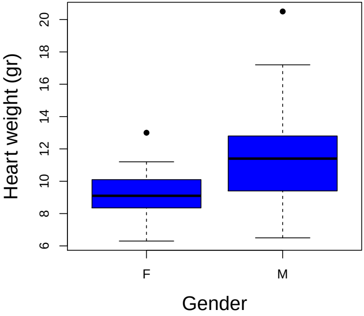

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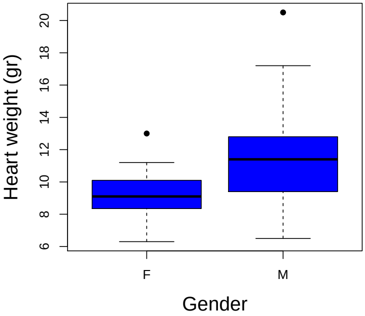
- For a male cat: $\hat{y}_i = \beta_0 + \beta_1$
- For a female cat: $\hat{y}_i = \beta_0$

$\Rightarrow$ β_1 quantifies the difference between female and male cats

Simple linear regression: an example

Is there a relationship between gender and a cat's heart weight?



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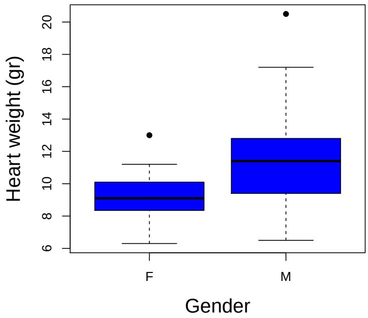

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Assuming $\epsilon_i \sim \text{Normal}(0, \sigma^2)$, we derive a t-test:

$$\beta_1 = 2.12$$
$$t = 5.35$$
$$p\text{-value} = 3.38 \times 10^{-7}$$

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$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_p x_{ip} + \epsilon_i$$

Multiple linear regression: an example

For example, with 2 continuous covariates:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where

- y_i: heart weight for cat i
- x_{i1}: body weight for cat i
- x_{i2}: age for cat i

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- **Global**: to assess the joint effect of all covariates

$$H_0 : \beta_1 = \cdots = \beta_p = 0 \text{ vs } H_1 : \text{at least one } \beta_j \neq 0$$

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Assuming $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$, we can use an *F*-test (from ANOVA table)

Multiple linear regression: an example

. . . or mixing a continuous and a categorical covariate

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where

- y_i: heart weight for cat i
- x_{i1}: body weight for cat i
- $x_{i2} = \begin{cases} 1, \text{if cat } i \text{ is male;} \\ 0, \text{if cat } i \text{ is female.} \end{cases}$

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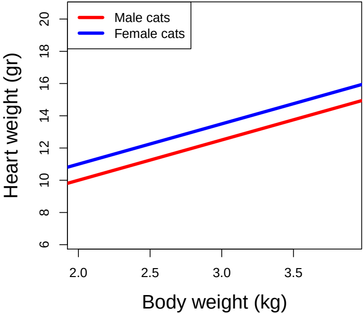

For a male cat:

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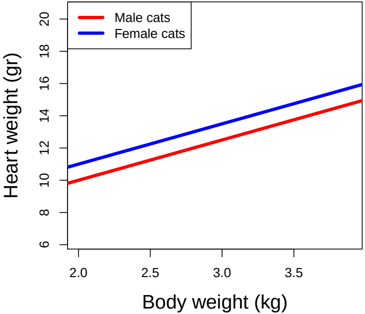

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⇒ parallel lines, effect of body weight is independent of gender

⇒ β_2 quantifies a global difference between female and male cats

Multiple linear regression: an example

What if we think the effect of body weight depends on gender?

Multiple linear regression: an example

What if we think the effect of body weight depends on gender?

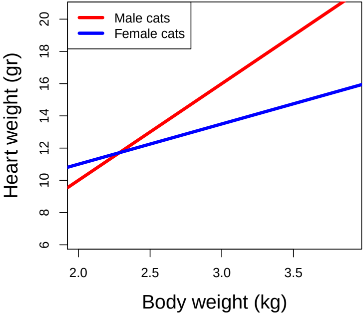
We can define an **interaction effect**

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i12} + \epsilon_i,$$

Recall:

- y_i: heart weight for cat i
- x_{i1}: body weight for cat i
- $x_{i2} = \begin{cases} 1, \text{if cat } i \text{ is male;} \\ 0, \text{if cat } i \text{ is female.} \end{cases}$

Multiple linear regression: an example



For a male cat:

$$\hat{y}_i = \beta_0 + \beta_1 x_{i1} + \beta_2 + \beta_3 x_{i1}$$

For a female cat:

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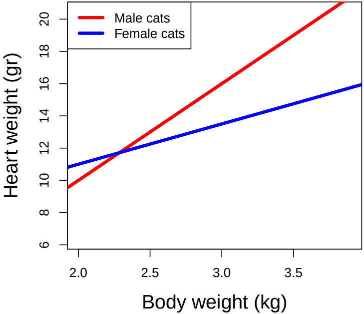

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$\Rightarrow$ crossing lines, effect of body weight depends on gender

$\Rightarrow$ β_2 quantifies a global difference between female and male cats

$\Rightarrow$ β_3 quantifies the differential effect of body weight

Questions + practical

Assumptions in linear regression

Assumptions in linear regression

Recall: the multiple linear regression model is written as

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_p x_{ip} + \epsilon_i$$

What assumptions underlie this model?

Assumptions in linear regression

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How can we assess this?

Assumptions in linear regression

Idea: after estimating the regression coefficients, define residuals as

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Use these estimated residuals to diagnose model quality

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More about this in the practical ...

Assumptions in linear regression

What if things go wrong?

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What if things go wrong?

If the residuals are not good enough to pass the diagnostic criteria, we need to revisit the model. For example

- We can transform some of the covariates or
- We can transform the response variable

Assumptions in linear regression: covariate transformation

Suppose a simple linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Covariate transformation can be useful in situations where the relationship between y and x is **not linear**

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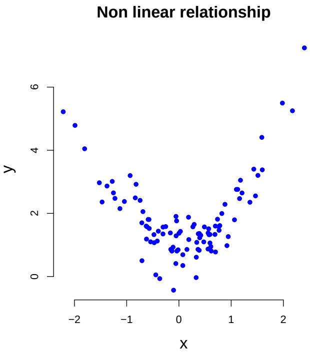

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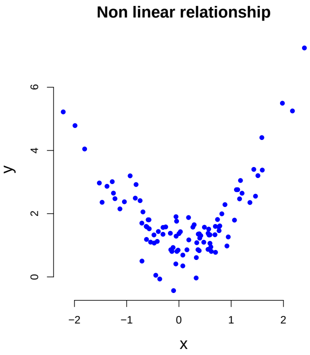

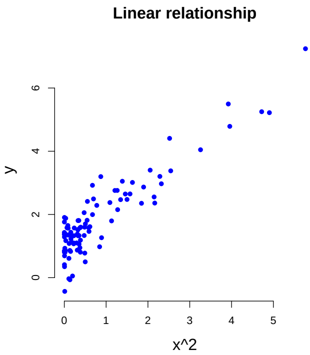

Assumptions in linear regression: response transformation

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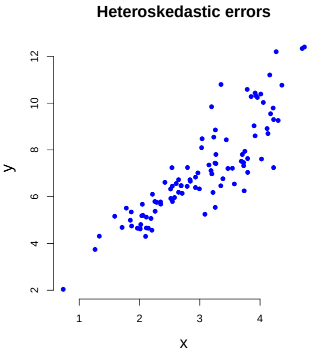

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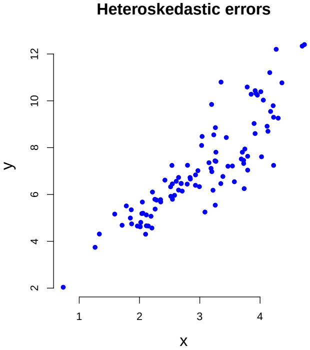

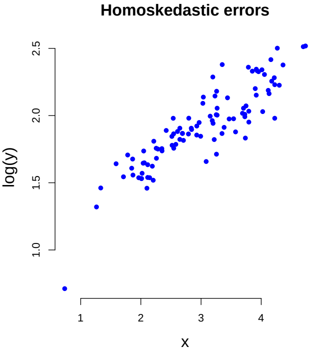

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- **Count-based response**

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We can use **Generalized Linear Models ...**

Questions + practical